BESSEL TYPE INEQUALITIES IN HILBERT C^*-MODULES

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Abstract. There are several well known generalizations of the Bessel inequality in Hilbert spaces. Among these we recall the ones due to Bombiari and Boas–Bellman. In this paper, we have obtained a version of the Bessel inequality and some generalizations of this inequality in the framework of Hilbert C^*-modules.

1. Introduction

The Bessel inequality states that if $(e_i)_{1\leq i\leq n}$ is an orthonormal family of vectors in a Hilbert space $(\mathscr{H};\langle .,.\rangle)$, then

$$\sum_{i=1}^{n}|\langle x,e_i\rangle|^2\leq\|x\|^2\qquad(x\in\mathscr{H}).$$

In the past, a number of mathematicians have investigated the above inequality in various settings. One of the generalizations of the Bessel inequality was given by Bombieri [3] as follows.

Theorem 1.1. *If $x,y_1,\cdots,y_n$ are elements of a complex unitary space, then*

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2\leq\|x\|^2\max_{1\leq i\leq n}\sum_{j=1}^{n}|\langle y_i,y_j\rangle|.$$

In 1941, Boas [2] and in 1944, independently, Bellman [1] proved a result can be stated as follows.

Theorem 1.2. *If $x,y_1,\cdots,y_n$ are elements of a Hilbert space, then*

$$\sum_{i=1}^{n}|\langle x,y_i\rangle|^2\leq\|x\|^2\left[\max_{1\leq i\leq n}\|y_i\|^2+\left(\sum_{1\leq i\neq j\leq n}|\langle y_i,y_j\rangle|^2\right)^{1/2}\right].$$

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Recently, Mitrinović–Pečarić–Fink [9] proved the following inequality and have shown that this inequality is equivalent with the Boas–Bellman theorem.

Theorem 1.3. *If $x, y_1, \cdots, y_n$ are elements of a Hilbert space and $c_1, \cdots, c_n$ are arbitrary complex numbers, then*

$$|\sum_{i=1}^{n} c_i\langle x, y_i\rangle|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left[\max_{1\leq i\leq n} \|y_i\|^2 + \left(\sum_{1\leq i\neq j\leq n} |\langle y_i, y_j\rangle|^2\right)^{1/2}\right].$$

In addition, Dragomir [4] obtained some other generalizations of the Bessel inequality, which are similar to Boas–Bellman inequality and Mitrinović–Pečarić–Fink inequality.

Our aim is to extend some of these generalizations of the Bessel inequality in the framework of Hilbert C^*-modules. A related notion to our work is that of frame. We would like to refer the interested reader to [6] for an extensive account of frames in Hilbert C^*-modules.

2. Preliminaries

In this section we recall some fundamental definitions in the theory of Hilbert modules that will be used in the sequel.

Suppose that $\mathscr{A}$ is a C^*-algebra and $\mathscr{X}$ is a linear space which is an algebraic right $\mathscr{A}$-module satisfying $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in \mathscr{X}, a \in \mathscr{A}, \lambda \in \mathbb{C}$.. The space $\mathscr{X}$ is called a *pre-Hilbert $\mathscr{A}$-module* (or an *inner product $\mathscr{A}$-module*) if there exists an *$\mathscr{A}$-valued inner product* $\langle ., . \rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ with the following properties:

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;

(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$;

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$;

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$;

for all $x, y, z \in \mathscr{X},\ a \in \mathscr{A},\ \lambda \in \mathbb{C}$.

By (ii) and (iv), $\langle \lambda x + ya, z\rangle = \bar{\lambda}\langle x, z\rangle + a^*\langle y, z\rangle$. It follows from the *Cauchy–Schwarz inequality* $\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle$ that $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ is a norm on $\mathscr{X}$, where the latter norm denotes that in the C^*-algebra $\mathscr{A}$; see [7, p. 5]. This norm makes $\mathscr{X}$ into a right normed module over $\mathscr{A}$. The pre-Hilbert module X is called a *Hilbert $\mathscr{A}$-module* if it is complete with respect to this norm.

Two typical examples of Hilbert C^*-modules are as follows:

(I) Every Hilbert space is a Hilbert $\mathbb{C}$-module.

(II) Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}$ is a Hilbert $\mathscr{A}$-module via $\langle a, b\rangle = a^*b \quad (a, b \in \mathscr{A})$. Notice that the inner product structure of a C^*-algebra is essentially more complicated than complex numbers. For instance, the concepts such as adjoint, orthogonality and theorems such as Riesz' representation in the complex Hilbert space theory cannot simply be generalized or transferred to the theory of Hilbert C^*-modules.

One may define an "*$\mathscr{A}$-valued norm*" $|.|$ by $|x| = \langle x, x\rangle^{1/2}$. Clearly, $\| \, |x| \, \| = \|x\|$ for each $x \in \mathscr{X}$. It is known that $|.|$ does not satisfy the triangle inequality in general. See [7, 8] for more detailed information on Hilbert C^*-modules.

3. MAIN RESULTS

We start our work by presenting a version of the Bessel inequality for Hilbert C^*-modules.

Theorem 3.1. *Let $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module and $e_1, e_2, \cdots, e_n$ be a family of unit vectors in X such that $\langle e_i, e_j\rangle = 0$ when $1 \le i \ne j \le n$. If $x \in X$, then*

$$\sum_{k=1}^{n} |\langle e_k, x\rangle|^2 \le |x|^2. \tag{3.1}$$

Proof. The result follows from the following inequalities.

$$\begin{aligned} 0 \le |x - \sum_{i=1}^{n} e_i\langle e_i, x\rangle|^2 &= \langle x - \sum_{i=1}^{n} e_i\langle e_i, x\rangle, x - \sum_{i=1}^{n} e_i\langle e_i, x\rangle\rangle \\ &= \langle x, x\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n} \langle e_i, x\rangle^*\langle e_i, e_j\rangle\langle e_j, x\rangle - 2\sum_{i=1}^{n} |\langle e_i, x\rangle|^2 \\ &= \langle x, x\rangle + \sum_{i=1}^{n} \langle e_i, x\rangle^*\langle e_i, e_i\rangle\langle e_i, x\rangle - 2\sum_{i=1}^{n} |\langle e_i, x\rangle|^2 \\ &\le |x|^2 + \sum_{i=1}^{n} \langle e_i, x\rangle^*\langle e_i, x\rangle - 2\sum_{i=1}^{n} |\langle e_i, x\rangle|^2 \\ &= |x|^2 - \sum_{i=1}^{n} |\langle e_i, x\rangle|^2. \end{aligned}$$

□

The following lemma is useful to prove a Bombieri type inequality.

Lemma 3.2. *Let $\mathscr{A}$ be a C^*-algebra and $a, b, c \in \mathscr{A}$, then*

$$a^*cb + b^*c^*a \leq \|c\|(|a|^2 + |b|^2).$$

Proof. Using the universal representation of $\mathscr{A}$ it is sufficient to prove that for bounded linear operators T, R, S acting on a Hilbert space $\mathscr{H}$, we have

$$T^*RS + S^*R^*T \leq \|R\|(|T|^2 + |S|^2).$$

By the polar decomposition, there exists a partial isometry $U \in \mathbb{B}(\mathscr{H})$ such that $R = U|R|$. It follows from

$$\begin{aligned} 0 &\leq (S - U^*T)^*|R|(S - U^*T) \\ &= S^*|R|S - S^*|R|U^*T - T^*U|R|S + T^*U|R|U^*T, \end{aligned}$$

that

$$T^*RS + S^*R^*T \leq |\,|R|^{1/2}U^*T|^2 + |\,|R|^{1/2}S|^2.$$

Hence

$$\begin{aligned} T^*RS + S^*R^*T &\leq |\,|R|^{1/2}U^*T|^2 + |\,|R|^{1/2}S|^2 \\ &\leq \|\,|R|^{1/2}U^*\|^2|T|^2 + \|\,|R|^{1/2}\|\,|S|^2 \\ &\leq \|R\|(|T|^2 + |S|^2). \end{aligned}$$

□

Theorem 3.3. *If $y_1, \cdots, y_n$ are elements of a Hilbert C^*-module $\mathscr{X}$, then*

$$|\sum_{i=1}^{n} y_i a_i|^2 \leq \sum_{i=1}^{n} |a_i|^2 \max_{1\leq i\leq n} \sum_{j=1}^{n} \|\langle y_i, y_j\rangle\|, \tag{3.2}$$

for all $a_1, \cdots, a_n \in \mathscr{A}$.

Proof. By using Lemma 3.2, we have

$$\begin{aligned}
|\sum_{i=1}^{n} y_i a_i|^2 &= \langle \sum_{i=1}^{n} y_i a_i, \sum_{i=1}^{n} y_i a_i \rangle = \sum_{1\le i,j\le n} a_i^* \langle y_i, y_j\rangle a_j \\
&= \sum_{i=1}^{n} a_i^* \langle y_i, y_i\rangle a_i + \sum_{1\le i\ne j\le n} a_i^* \langle y_i, y_j\rangle a_j \\
&= \sum_{i=1}^{n} a_i^* \langle y_i, y_i\rangle a_i + \sum_{1\le i< j\le n} (a_i^* \langle y_i, y_j\rangle a_j + a_j^* \langle y_j, y_i\rangle a_i) \\
&\le \sum_{i=1}^{n} \|\langle y_i, y_i\rangle\| \, |a_i|^2 + \sum_{1\le i< j\le n} (\|\langle y_i, y_j\rangle\| \, |a_i|^2 + \|\langle y_i, y_j\rangle\| \, |a_j|^2) \\
&= \sum_{1\le i,j\le n} \|\langle y_i, y_j\rangle\| \, |a_i|^2 \\
&\le \sum_{i=1}^{n} |a_i|^2 \max_{1\le i\le n} \sum_{j=1}^{n} \|\langle y_i, y_j\rangle\| \,.
\end{aligned}$$

□

From Theorem 3.3 the following result of Bombieri type can be obtained.

Corollary 3.4. *Let $x, y_1, \cdots, y_n \in \mathscr{X}$ and $a_1, \cdots, a_n \in \mathscr{A}$, then it holds*

$$\left(\sum_{i=1}^{n} |\langle y_i, x\rangle|^2\right)^2 \le |x|^2 \left\| \sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \right\| \max_{1\le i\le n} \sum_{j=1}^{n} \|\langle y_i, y_j\rangle\| \,.$$

Proof. By using inequality (3.2), we have

$$\begin{aligned}
\left(\sum_{i=1}^{n} |\langle y_i, x\rangle|^2\right)^2 &= \left| \sum_{i=1}^{n} \langle y_i, x\rangle^* \langle y_i, x\rangle \right|^2 \\
&= \left| \langle \sum_{i=1}^{n} y_i \langle y_i, x\rangle, x\rangle \right|^2 \\
&\le \left\| \sum_{i=1}^{n} y_i \langle y_i, x\rangle \right\|^2 |x|^2 \\
&\le |x|^2 \left\| \sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \right\| \max_{1\le i\le n} \sum_{j=1}^{n} \|\langle y_i, y_j\rangle\| \\
&\le |x|^2 \left\| \sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \right\| \max_{1\le i\le n} \sum_{j=1}^{n} \|\langle y_i, y_j\rangle\| \,.
\end{aligned}$$

□

Now we state some other applications of inequality (3.2) in Hilbert space operators, although some of them can be deduced directly. Recall that the space $\mathbb{B}(\mathscr{H},\mathscr{K})$ of all bounded linear operators between Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ can be regarded as a Hilbert C^*-module over the C^*-algebra $\mathbb{B}(\mathscr{H})$ via $\langle T,S\rangle = T^*S$.

Corollary 3.5. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and $T_1,\cdots,T_n \in \mathbb{B}(\mathscr{H},\mathscr{K})$ be operators having orthogonal ranges. Then*

$$\left|\sum_{i=1}^n T_iS_i\right|^2 \leq \sum_{i=1}^n |S_i|^2 \max_{1\leq i\leq n} \|T_i\|^2,$$

for any $S_1,\cdots,S_n \in \mathbb{B}(\mathscr{H})$.

Proof. Clearly the operators T_i have orthogonal ranges if and only if $T_i^*T_j = 0$. Thus the result follows immediately from inequality (3.2). □

Corollary 3.6. *Let S_1, S_2 be two operators on a Hilbert space $\mathscr{H}$ and T be an invertible operator on $\mathscr{H}$. Then*

$$|TS_1 + (T^*)^{-1}S_2|^2 \leq (|S_1|^2 + |S_2|^2)\left[1 + \max(\|T\|^2, \|T^{-1}\|^2)\right].$$

The next result is a refinement of one of the inequalities, which are given in Theorem 2.1 of [5].

Corollary 3.7. *If $\lambda_1,\cdots\lambda_n$ are complex numbers and $T_1,\cdots,T_n$ are operators on a Hilbert space $\mathscr{H}$, then*

$$|\sum_{i=1}^n \lambda_iT_i| \leq \max_{1\leq i\leq n} |\lambda_i| \sum_{i=1}^n |\lambda_i| \sum_{i=1}^n |T_i|^2.$$

Proof. If we consider $\lambda_iT_i = (\lambda_iI)T_i$, we get from (3.2) that

$$\begin{aligned}\left|\sum_{i=1}^n \lambda_iT_i\right| &\leq \sum_{i=1}^n |T_i|^2 \max_{1\leq i\leq n} \sum_{j=1}^n |\bar{\lambda}_i\lambda_j| \\ &= \sum_{i=1}^n |T_i|^2 \max_{1\leq i\leq n} |\lambda_i| \sum_{j=1}^n |\lambda_j| \\ &= \max_{1\leq i\leq n} |\lambda_i| \sum_{i=1}^n |\lambda_i| \sum_{i=1}^n |T_i|^2.\end{aligned}$$

□

The following theorem is similar to the Mitrinović–Pečarić–Fink theorem in the Hilbert space theory, with $\mathscr{A}$-valued norm instead of usual norm.

Theorem 3.8. *If $x, y_1, \cdots, y_n$ are elements of a Hilbert $\mathscr{A}$-module $\mathscr{X}$ and $a_1, \cdots, a_n$ are elements of $\mathscr{A}$, then*

$$\left|\sum_{i=1}^{n} a_i\langle y_i, x\rangle\right|^2 \le |x|^2 \sum_{i=1}^{n} \|a_i\|^2 \left[\max \|y_i\|^2 + \left(\sum_{1\le i\ne j\le n} \|\langle y_i, y_j\rangle\|^2\right)^{1/2}\right]. \tag{3.3}$$

Proof. By the Cauchy–Schwarz inequality,

$$\left|\sum_{i=1}^{n} a_i\langle y_i, x\rangle\right|^2 = \left|\langle\sum_{i=1}^{n} y_i a_i^*, x\rangle\right|^2 \le \left\|\sum_{i=1}^{n} y_i a_i^*\right\|^2 |x|^2. \tag{3.4}$$

We have

$$\begin{aligned}
\left\|\sum_{i=1}^{n} y_i a_i^*\right\|^2 &= \left\|\langle\sum_{i=1}^{n} y_i a_i^*, \sum_{j=1}^{n} y_j a_j^*\rangle\right\| \\
&= \left\|\sum_{1\le i,j\le n} a_i\langle y_i, y_j\rangle a_j^*\right\| \\
&\le \sum_{1\le i,j\le n} \|a_i\|\|\langle y_i, y_j\rangle\|\|a_j\| \\
&= \sum_{i=1}^{n} \|a_i\|^2\|y_i\|^2 + \sum_{1\le i\ne j\le n} \|a_i\|\|a_j\|\|\langle y_i, y_j\rangle\| \\
&\le \max_{1\le i\le n} \|y_i\| \sum_{i=1}^{n} \|a_i\|^2 + \left(\sum_{1\le i\ne j\le n} \|a_i\|^2\|a_j\|^2\right)^{\frac{1}{2}} \left(\sum_{1\le i\ne j\le n} \|\langle y_i, y_j\rangle\|^2\right)^{\frac{1}{2}} \\
&\le \sum_{i=1}^{n} \|a_i\|^2 \left(\max_{1\le i\le n} \|y_i\|^2 + \left(\sum_{1\le i\ne j\le n} \|\langle y_i, y_j\rangle\|^2\right)^{\frac{1}{2}}\right). \qquad (3.5)
\end{aligned}$$

Combining (3.4) and (3.5), we can get the desired result (3.3). □

Corollary 3.9. *For $x, y_1, \cdots, y_n$ in a Hilbert $\mathscr{A}$-module $\mathscr{X}$,*

$$\left(\sum_{i=1}^{n} |\langle y_i, x\rangle|^2\right)^2 \le |x|^2 \sum_{i=1}^{n} \|\langle y_i, x\rangle\|^2 \left[\max_{1\le i\le n} \|y_i\|^2 + \left(\sum_{1\le i\ne j\le n} \|\langle y_i, y_j\rangle\|^2\right)^{\frac{1}{2}}\right]. \tag{3.6}$$

Proof. Set $a_i = \langle x, y_i \rangle$, in (3.3). □

Note that the inequality (3.6), can be considered as a generalization of Boas–Bellman inequality (1.1). However, for the case where $(y_i)_{1\le i\le n}$ is an orthonormal family of vectors, the inequality (3.6) is a weaker result than (3.1).

Also all of the inequalities which are obtained by Dragomir in [4, Ch. 4], can be extended to Hilbert C^*-modules in a similar way. The details are left to the interested readers.

We can prove some other Boas–Bellman type inequalities in Hilbert C^*-modules as follows.

Lemma 3.10. *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module. Then*

$$|\sum_{i=1}^n y_i a_i|^2 \le \max_{1\le i\le n} \|y_i\|^2 \sum_{i=1}^n |a_i|^2 + B_n,$$

where

$$B_n = \begin{cases} (n-1)\sqrt{n} \max\limits_{1\le i\le n} \|a_i\| \max\limits_{1\le i\ne j\le n} \|\langle y_i, y_j\rangle\| \left(\sum\limits_{i=1}^n |a_i|^2\right)^{\frac{1}{2}}, \\ \sqrt{n-1}\left(\max\limits_i \sum\limits_{1\le j\ne i\le n} \|\langle y_i, y_j\rangle\|^2\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^n \|a_i\|^2\right)^{\frac{1}{2}} \left(\sum\limits_{i=1}^n |a_i|^2\right)^{\frac{1}{2}}, \end{cases} \tag{3.7}$$

for any $y_i \in \mathscr{X}$ and $a_i \in \mathscr{A}$.

Proof. We observe that

$$\begin{aligned} |\sum_{i=1}^n y_i a_i|^2 &= \langle \sum_{i=1}^n y_i a_i, \sum_{i=1}^n y_i a_i \rangle \\ &= \sum_{1\le i,j\le n} a_i^* \langle y_i, y_j \rangle a_j \\ &= \sum_{i=1}^n a_i^* \langle y_i, y_i \rangle a_i + \sum_{1\le i\ne j\le n} a_i^* \langle y_i, y_j \rangle a_j. \end{aligned}$$

We also have

$$\sum_{i=1}^n a_i^* \langle y_i, y_i \rangle a_i = \sum_{i=1}^n a_i^* |y_i|^2 a_i \le \sum_{i=1}^n \|y_i\|^2 |a_i|^2 \le \max_{1\le i\le n} \|y_i\|^2 \sum_{i=1}^n |a_i|^2.$$

To get the first inequality of (3.7) note that

$$\begin{aligned}
\sum_{1\le i\ne j\le n} a_i^*\langle y_i, y_j\rangle a_j &= 2\sum_{1\le i<j\le n} \mathrm{Re}(a_i^*\langle y_i, y_j\rangle a_j) \\
&\le 2\sum_{1\le i<j\le n}\left(\frac{|a_i^*\langle y_i, y_j\rangle a_j|^2 + |a_j^*\langle y_j, y_i\rangle a_i|^2}{2}\right)^{1/2} \\
&\qquad\qquad \left(\text{by } |\mathrm{Re}(c)| \le \left(\frac{c^*c + cc^*}{2}\right)^{1/2},\ \text{where } c\in\mathscr{A}\right) \\
&= \sqrt{2}\sum_{1\le i<j\le n}\left(|a_i^*\langle y_i, y_j\rangle a_j|^2 + |a_j^*\langle y_j, y_i\rangle a_i|^2\right)^{1/2}.
\end{aligned}$$

By operator convexity of the function $f(t) = t^2$, we conclude that

$$\begin{aligned}
\sum_{i\ne j} a_i^*\langle y_i, y_j\rangle a_j &\le \sqrt{2}.\sqrt{\frac{n^2-n}{2}}\left(\sum_{1\le i<j\le n} |a_i^*\langle y_i, y_j\rangle a_j|^2 + |a_j^*\langle y_j, y_i\rangle a_i|^2\right)^{1/2} \\
&= \sqrt{n^2-n}\left(\sum_{1\le i\ne j\le n} |a_i^*\langle y_i, y_j\rangle a_j|^2\right)^{1/2} \\
&\le \sqrt{n^2-n}\max_{1\le i\le n}\|a_i\|\max_{1\le i\ne j\le n}\|\langle y_i, y_j\rangle\|\left((n-1)\sum_{i=1}^n |a_i|^2\right)^{1/2} \\
&= (n-1)\sqrt{n}\max_{1\le i\le n}\|a_i\|\max_{1\le i\ne j\le n}\|\langle y_i, y_j\rangle\|\left(\sum_{i=1}^n |a_i|^2\right)^{1/2}.
\end{aligned}$$

To obtain the second inequality of (3.7) notice that $\sum_{1\leq i\neq j\leq n} a_i^*\langle y_i, y_j\rangle a_j$ is self-adjoint, so that

$$\begin{aligned}
\left(\sum_{1\leq i\neq j\leq n} a_i^*\langle y_i, y_j\rangle a_j\right)^2 &\leq \left|\sum_{1\leq i\neq j\leq n} a_i^*\langle y_i, y_j\rangle a_j\right|^2 \\
&= \left|\sum_{i=1}^n a_i^*\left(\sum_{1\leq j\neq i\leq n} \langle y_i, y_j\rangle a_j\right)\right|^2 \\
&\leq \sum_{i=1}^n \|a_i\|^2 \sum_{i=1}^n \left|\sum_{1\leq j\neq i\leq n} \langle y_i, y_j\rangle a_j\right|^2 \\
&\leq \sum_{i=1}^n \|a_i\|^2 \sum_{i=1}^n \left(\sum_{1\leq j\neq i\leq n} \|\langle y_i, y_j\rangle\|^2 \sum_{1\leq j\neq i\leq n} |a_j|^2\right) \\
&\leq \sum_{i=1}^n \|a_i\|^2 \left(\max_{1\leq i\leq n} \sum_{1\leq j\neq i\leq n} \|\langle y_i, y_j\rangle\|^2 . \sum_{i=1}^n \sum_{1\leq j\neq i\leq n} |a_j|^2\right) \\
&\leq (n-1)\sum_{i=1}^n |a_i|^2 \left(\sum_{i=1}^n \|a_i\|^2 . \max_{1\leq i\leq n} \sum_{1\leq j\neq i\leq n} \|\langle y_i, y_j\rangle\|^2\right).
\end{aligned}$$

□

The following result can be stated as well.

Theorem 3.11. *Let $x, y_1, \cdots, y_n \in \mathscr{X}$ and $a_1, \cdots, a_n \in \mathscr{A}$. Then*

$$\left|\sum_{i=1}^n a_i\langle y_i, x\rangle\right|^2 \leq |x|^2 \left\|\sum_{i=1}^n |a_i^*|^2\right\|^{1/2} \left[\max_{1\leq i\leq n} \|y_i\|^2 \|\sum_{i=1}^n |a_i^*|^2\|^{1/2} + B_n\right],$$

where

$$B_n := \begin{cases} (n-1)\sqrt{n}\max_{1\leq i\leq n}\|a_i\| \max_{1\leq i\neq j\leq n} \|\langle y_i, y_j\rangle\|, \\ \sqrt{n-1}\left(\max_{1\leq i\leq n} \sum_{1\leq j\neq i\leq n} \|\langle y_i, y_j\rangle\|^2\right)^{1/2} \left(\sum_{i=1}^n \|a_i\|^2\right)^{1/2}. \end{cases}$$

Remark 3.12. In the case where (y_i) are orthogonal (but not necessarily $\|y_i\| = 1$), it follows from the theorem above that

$$\left(\sum_{i=1}^n |\langle y_i, x\rangle|^2\right)^2 \leq |x|^2 \max_{1\leq i\leq n} \|y_i\|^2 \left\|\sum_{i=1}^n |\langle y_i, x\rangle|^2\right\|,$$

which is a stronger result than the inequality (3.6).

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