ON TWO- AND FOUR-PARAMETER FAMILIES

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Abstract. We investigate monotonicity and convexity properties of the two-parameter function of the form

$$\mathcal{H}_f(p,q;x,y) = \left(\frac{f(x^p,y^p)}{f(x^q,y^q)}\right)^{1/(p-q)}.$$

1. Introduction

Let $f : \mathbf{R}_+^2 \to \mathbf{R}_+$ be a symmetric and positively homogeneous function (i.e. for $\lambda > 0$ $f(\lambda x, \lambda y) = \lambda f(x,y)$), satisfying $f(1,1) = 1$. For real p,q we define the function

$$\mathcal{H}_f(p,q;x,y) = \begin{cases} \left(\dfrac{f(x^p,y^p)}{f(x^q,y^q)}\right)^{1/(p-q)} & p \neq q, \\ \exp(\frac{d}{dp}\log f(x^p,y^p)) & p = q \neq 0, \\ \sqrt{xy} & p = q = 0. \end{cases} \tag{1.1}$$

We call $\mathcal{H}_f$ the two-parameter family generated by f. In 2005 Zhen-Hang Yang published series of preprints ([7, 8, 9, 10, 11]) investigating monotonicity and logarithmic convexity of $\mathcal{H}_f$. He showed that the sign of $(\log f)_{xy}$ is responsible for monotonicity of $\mathcal{H}_f$ in p and q, while $(x-y)(x(\log f)_{xy})_x$ decides the logarithmic convexity along some horizontal and vertical half-lines in the space (p,q).
This note extends the results of Yang, simplifies proofs and gives other conditions equivalent to monotonicity and convexity of $\mathcal{H}_f$. As a corollary we obtain some inequalities between Stolarsky, Heronian and Gini means.
We also investigate four-parameter families being iteration of the procedure (1.1).

While Yang uses straightforward differentiations to investigate convexity and monotonicity properties, we chose a different approach. Two functions will play an important role: $\widetilde{f}(t) = f(t,1)$ and $\widehat{f}(t) = \log \widetilde{f}(\exp(t))$. Due to homogeneity of f the identity

$$\widetilde{f}(t) = t\widetilde{f}(1/t) \tag{1.2}$$

holds for all positive t. Note that the formula $y\widetilde{f}(x/y) = f(x,y)$ gives 1-1 correspondence between homogeneous functions f and functions satisfying (1.2).

The function $\widehat{f}$ is important due to the following identity:

$$\mathcal{H}_f(p,q;x,y) = y\exp\frac{\widehat{f}(p\log(x/y)) - \widehat{f}(q\log(x/y))}{p-q} \tag{1.3}$$

which allows to express the properties of $\mathcal{H}_f$ by those of $\widehat{f}$.

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Replacing t by e^t in (1.2) and differentiating we obtain the formulas

$$\widehat{f}(t) = t + \widehat{f}(-t) \tag{1.4}$$

$$\widehat{f}\,'(t) - 1/2 = 1/2 - \widehat{f}\,'(-t) \tag{1.5}$$

$$\widehat{f}\,''(t) = \widehat{f}\,''(-t) \tag{1.6}$$

$$\widehat{f}\,'''(t) = -\widehat{f}\,'''(-t) \tag{1.7}$$

The identities below follow immediately from definition

$$\mathcal{H}_f(p,-p;x,y) = \sqrt{xy}, \tag{1.8}$$

$$\mathcal{H}_f(p,q;x^a,y^a) = \mathcal{H}_f^a(ap,aq;x,y), \tag{1.9}$$

$$\mathcal{H}_f(-p,-q;x,y) = \frac{xy}{\mathcal{H}_f(p,q;x,y)}. \tag{1.10}$$

The last formula can be also written as

$$\log \mathcal{H}_f(-p,-q;x,y) = \log(xy) - \log \mathcal{H}_f(p,q;x,y) \tag{1.11}$$

and generalized as follows:

Lemma 1.1. *For $p+q \neq 0$*

$$\left[\frac{\mathcal{H}_f(p,q;x,y)}{\sqrt{xy}}\right]^{\frac{1}{p+q}} = \left[\frac{\mathcal{H}_f(|p|,|q|;x,y)}{\sqrt{xy}}\right]^{\frac{1}{|p|+|q|}}.$$

Proof. For $p,q>0$ the lemma is obvious, case $p,q<0$ follows from identity (1.11), so let us assume that $q<0\le p$. We have

$$\begin{aligned}\mathcal{H}_f(p,q;x,y) &= \left(\frac{f(x^p,y^p)}{f(x^q,y^q)}\right)^{1/(p-q)} = \left(\frac{f(x^p,y^p)}{(xy)^q f(x^{|q|},y^{|q|})}\right)^{1/(p-q)} = \\ &= (xy)^{\frac{-q}{|p|+|q|}} \left(\frac{f(x^{|p|},y^{|p|})}{f(x^{|q|},y^{|q|})}\right)^{1/(|p|+|q|)} \\ &= (xy)^{\frac{|p|+|q|-(p+q)}{2(|p|+|q|)}} \left(\mathcal{H}_f(|p|,|q|;x,y)\right)^{\frac{p+q}{|p|+|q|}}.\end{aligned}$$

□

2. Monotonicity

In this section we will discuss the monotonicity of $\mathcal{H}_f$. Taking $f(x,y) = \frac{x+y}{2}$ we see that although f is increasing, $\mathcal{H}_f(2,1;x,y) = \frac{x^2+y^2}{x+y}$ is not, so we need something more to grant monotonicity in x and y. But this property is sufficient for $\mathcal{H}_f$ to be a mean:

Theorem 2.1. *The following conditions are equivalent:*

(a) *f is increasing in both variables.*
(b) *$\widetilde{f}$ is increasing.*
(c) *$\widehat{f}$ is increasing.*
(d) *for all p,q $\mathcal{H}_f$ is a mean, i.e. for all $x<y$*

$$x \le \mathcal{H}_f(p,q;x,y) \le y.$$

Proof. The equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) is obvious.
(a) $\Rightarrow$ (d): due to symmetry we can assume that $p > q$. We have

$$x^{p-q} = \frac{f(x^q x^{p-q}, y^q x^{p-q})}{f(x^q, y^q)} \leq \frac{f(x^p, y^p)}{f(x^q, y^q)} \leq \frac{f(x^q y^{p-q}, y^q y^{p-q})}{f(x^q, y^q)} = y^{p-q}.$$

(d) $\Rightarrow$ (b) Let $x < y$.
If $1 < x$ then $y = x^p$ for some $p > 1$ and this yields

$$\frac{f(y,1)}{f(x,1)} = \frac{f(x^p,1)}{f(x,1)} = \mathcal{H}_f^{p-1}(p,1;x,1) > 1,$$

similarly if $x < 1$ then $y = x^p$ for some $p < 1$ and the same inequality holds. □

The two theorems that follow state the necessary and sufficient conditions for $\mathcal{H}_f$ to be monotone in p, q and x, y respectively.

Theorem 2.2. *The following conditions are equivalent*

(a) *(The Hölder inequality). If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x_1, x_2, y_1, y_2 > 0$*

$$f(x_1 x_2, y_1 y_2) \leq f^{1/p}(x_1^p, y_1^p) f^{1/q}(x_2^q, y_2^q)$$

(b) *The function*

$$G(u,v) = \log f(e^u, e^v)$$

is convex.

(c) *For every $x, y > 0$ the function*

$$T(p) = \log f(x^p, y^p)$$

is convex.

(d) *$\widetilde{f}$ is multipicatively convex, i.e. for every $0 < \lambda < 1$*

$$\widetilde{f}(x^\lambda y^{1-\lambda}) \leq \left[\widetilde{f}(x)\right]^\lambda \left[\widetilde{f}(y)\right]^{1-\lambda}.$$

(e) *$\widehat{f}$ is convex*

(f) *The function $\mathcal{H}_f(p,q;x,y)$ increases in p and q.*

Note: the result of Yang states that if $(\log f)_{xy} \geq 0$ then 2.2(f) holds. In fact, the Yang's condition is equivalent to $T''(p) \geq 0$.

Proof.

(a)$\Leftrightarrow$(b) Set $\exp(u_i) = x_i^p,\ \exp(v_i) = y_i^p$.

(d)$\Leftrightarrow$(e) obvious.

(e)$\Leftrightarrow$(f) h is convex (concave) if and only if the divided difference function $\frac{h(p)-h(q)}{p-q}$ is increasing (decreasing) in p and q [12]. By (1.3)

$$\log \mathcal{H}_f(p,q;x,y) = \log y + \log(x/y)\frac{\widehat{f}(p\log(x/y)) - \widehat{f}(q\log(x/y))}{p\log(x/y) - q\log(x/y)},$$

hence the assertion follows.

(b)$\Rightarrow$(c)

$$T(p) = G(p\log x, p\log y).$$

(c)$\Leftrightarrow$(e) follows from the identity

$$T(p) = p\log y + \log f((x/y)^p, 1) = p\log y + \widehat{f}(p\log(x/y))$$

(d)⇒(a)

$$\begin{aligned} f(x_1x_2, y_1y_2) &= y_1y_2\widetilde{f}\left(\left(x_1^p/y_1^p\right)^{1/p}\left(x_2^q/y_2^q\right)^{1/q}\right) \\ &\le y_1y_2\widetilde{f}^{1/p}\left(x_1^p/y_1^p\right)\widetilde{f}^{1/p}\left(x_2^q/y_2^q\right) = f^{1/p}(x_1^p, y_1^p)f^{1/q}(x_2^q, y_2^q) \end{aligned}$$

□

A homogeneous positive symmetric function cannot decrease in it's whole domain because it satisfies the identity $f(x, x) = xf(1, 1)$. Thus if it is monotone then it has to increase.

Theorem 2.3. *For every p, q the function $\mathcal{H}_f(p, q; x, y)$ is increasing in x and y if and only if the function $t\widehat{f}\,'(t)$ is increasing .*

Proof. Due to homogeneity and symmetry of $\mathcal{H}_f$ in x and y if is enough to prove the theorem in case $y = 1$.
The monotonicity of $\mathcal{H}_f(p, q; x, 1)$ is the same as that of $\log \mathcal{H}_f(p, q; \exp(x), 1)$. Differentiating we obtain by (1.3)

$$\frac{d\log\mathcal{H}_f(p,q;\exp(t),1)}{dt} = \frac{p\widehat{f}\,'(pt) - q\widehat{f}\,'(qt)}{p-q} \tag{2.1}$$

$$= \frac{pt\widehat{f}\,'(pt) - qt\widehat{f}\,'(qt)}{pt-qt}. \tag{2.2}$$

The divided difference (2.2) preserves sign if and only if the function $t\widehat{f}\,'(t)$ is monotone and the proof is complete. □

If $\widehat{f}\,'(t)$ is nonnegative and $pq \le 0$ then the numerator and the denominator of (2.1) are of the same sign, so we have

Corollary 2.4. *If f is increasing and $pq \le 0$ then $\mathcal{H}_f(p, q; x, y)$ is increasing in x and y.*

Note the following necessary condition for monotonicity in x, y:

Theorem 2.5. *If for every p, q $\mathcal{H}_f(p, q; x, y)$ is increasing in x and y then $f(x, y) = \max(x, y)$ or $\lim_{x\to 0}\widetilde{f}(t) = 0$.*

Proof. The limit of $\widetilde{f}$ at 0 exists because of monotonicity. If it is positive then for positive $p \ne q$

$$\lim_{x\to 0}\mathcal{H}_f(p,q;x,1) = \lim_{x\to 0}\left(\frac{\widetilde{f}(x^p)}{\widetilde{f}(x^q)}\right)^{1/(p-q)} = 1 = \mathcal{H}_f(p,q;1,1)$$

and this is possible only if $\widetilde{f}$ is constant on $(0, 1)$ which corresponds to $f = \max$. □

We conclude this section with some kind of Chebyhshev's inequality:

Corollary 2.6. *If $\widehat{f}$ is convex then the inequality*

$$f(x_1, y_1)f(x_2, y_2) \le (\text{ resp. } \ge) f(x_1x_2, y_1y_2) \tag{2.3}$$

holds if and only if

$$(x_1 - y_1)(x_2 - y_2) \ge (\text{ resp. } \le) 0. \tag{2.4}$$

For concave $\widehat{f}$ the inequality in (2.3) reverses.

Proof. Let $a = x_1/y_1, b = x_2/y_2$. Then $(x_1 - y_1)(x_2 - y_2) \geq (\leq)0$ holds if and only if there exists $p > (<)0$ such that $b = a^p$. By Theorem 2.2

$$f(a, 1) = \mathcal{H}_f(0, 1; a, 1) \leq (\geq)\mathcal{H}_f(p, p + 1; a, 1) = \frac{f(ab, 1)}{f(b, 1)}$$

and this is equivalent to (2.3). □

3. Logarithmic convexity

In this section we will cover the log-convexity of $\mathcal{H}_f$ in variables p and q. The identity (1.11) shows that concavity of $\log \mathcal{H}_f$ at some point implies convexity at its antipode. Milan Merkle [3] discovered the following characterization of convexity of divided difference functions:

Theorem 3.1. *Let $f : I \to \mathbf{R}$ be differentiable and*

$$F(p, q) = \begin{cases} \dfrac{f(p) - f(q)}{p - q} & p \neq q, \\ f'(p) & p = q. \end{cases}$$

. Then the following conditions are equivalent:

(a) *f' is convex on I,*
(b) *$f'\left(\frac{p+q}{2}\right) \leq F(p, q)$ for all $p, q \in I$,*
(c) *$F(p, q) \leq \dfrac{f'(p) + f'(q)}{2}$ for all $p, q \in I$,*
(d) *F is convex on I^2,*
(e) *F is Schur-convex on I^2.*

The eqivalence remains valid if the word 'convex' is replaced with 'concave' and inequalities in (b) and (c) are reversed.

Suppose now that $I \subset \mathbf{R}_+$ and $\log \mathcal{H}_f$ is convex in p, q for all $x, y > 0$. Using the representation (1.3) and Theorem 3.1 we see that $\frac{d\widehat{f}(p\log(x/y))}{dp} = \log(x/y)\widehat{f}\,'(p\log(x/y))$ must be convex on I. Because $\log(x/y)$ takes arbitrary values, this is possible only if $\widehat{f}\,'$ is convex on $\mathbf{R}_+$ and concave otherwise.
On the other hand (1.5) shows that convexity (concavity) of $\widehat{f}\,'$ on $(0, \infty)$ implies its concavity (convexity) on $(-\infty, 0)$. Hence we have

Theorem 3.2. *The following conditions are equivalent:*

(a) *For all $p, q \geq 0$ and all $x, y > 0$ $\log \mathcal{H}_f$ is convex (concave) in p and q.*
(b) *For all $p, q \geq 0$ and all $x, y > 0$ $\log \mathcal{H}_f$ is Schur-convex (Schur-concave) in p and q.*
(c) *$\widehat{f}\,'(t)$ is convex (concave) for $t \geq 0$.*
(d) *For all $p, q \leq 0$ and all $x, y > 0$ $\log \mathcal{H}_f$ is concave (convex) in p and q.*
(e) *For all $p, q \leq 0$ and all $x, y > 0$ $\log \mathcal{H}_f$ is Schur-concave (Schur-convex) in p and q.*
(f) *$\widehat{f}\,'(t)$ is concave (convex) for $t \leq 0$.*

Before we investigate how $\mathcal{H}_f$ behaves along some straight lines in (p, q) we formulate an useful lemma:

Lemma 3.3. *Let $f : \mathbf{R} \to \mathbf{R}$ be an even function. Then f is strictly increasing in $(0, \infty)$ if and only if for all a, b*

$$\operatorname{sgn} \frac{f(a) - f(b)}{a - b} = \operatorname{sgn}(a + b) \tag{3.1}$$

and strictly decreasing if and only if

$$\operatorname{sgn} \frac{f(a) - f(b)}{a - b} = -\operatorname{sgn}(a + b) \tag{3.2}$$

Proof.

$$\frac{f(a) - f(b)}{a - b} = (a + b)\frac{f(|a|) - f(|b|)}{a^2 - b^2} = (a + b)\frac{|a| - |b|}{a^2 - b^2}\frac{(f|a|) - f(|b|)}{|a| - |b|}$$

and the lemma follows because $\operatorname{sgn} \frac{|a|-|b|}{a^2-b^2} = 1$. □

Consider first the covexity on lines passing through the origin.

Theorem 3.4. *Let $\widehat{f}\,'(t)$ be concave (convex) for $t \geq 0$. Then for $p + q > 0$*

$$h(t) = \log \mathcal{H}_f(tp, tq; x, y)$$

is concave (convex) for $t \geq 0$ and convex (concave) for $t \leq 0$. The convexity reverses if $p + q < 0$.

Proof. By Lemma 1.1 we have

$$\log \mathcal{H}_f(tp, tq; x, y) = \frac{|p| + |q| - (p + q)}{|p| + |q|} \log \sqrt{xy} + \frac{p + q}{|p| + |q|} \log \mathcal{H}_f(t|p|, t|q|; x, y)$$

and the theorem follows from Theorem 3.2. □

A concave function that is bounded in $+\infty$ must be increasing. The same applies to a convex function bounded in $-\infty$. If $\mathcal{H}_f$ is a mean then obviously h is bounded, so we have

Corollary 3.5. *If $\widehat{f}\,'(t)$ is concave for $t \geq 0$ and $\widehat{f}(t)$ is increasing then $h(t)$ is increasing.*

Consider now lines that are parallel to the diagonal. The theorem that follows generalizes results obtained by Horst Alzer [1, 2] and the author [13].

Theorem 3.6. *Let $\widehat{f}\,'(t)$ be concave (convex) for $t \geq 0$. Then*

$$S_h(t) = \mathcal{H}_f(t + h, t; x, y)$$

is log-concave (log-convex) for $t \geq -h/2$ and log-convex (log-concave) for $t \leq -h/2$.

Proof. By (1.3) we have

$$(\log S_h)''(t) = \log^3(x/y)\frac{\widehat{f}\,''((t + h)\log(x/y)) - \widehat{f}\,''(t\log(x/y))}{(t + h)\log(x/y) - t\log(x/y)}$$

and the assertion follows from (1.6), (1.7) and Lemma 3.3. □

Applying the same reasoning as before we obtain

Corollary 3.7. *If $\widehat{f}\,'(t)$ is concave for $t \geq 0$ and $\widehat{f}(t)$ is increasing then $S_h(t)$ is increasing.*

Finally let us consider lines perpendicular to the diagonal:

Theorem 3.8. *Let $\widehat{f}\,'(t)$ be concave (convex) for $t \geq 0$. For $a > 0$ the even function*

$$v_a(r) = \mathcal{H}_f(a+r, a-r; x, y)$$

is decreasing (increasing) for $r > 0$. The monotonicity reverses if $a < 0$.

Proof. In the proof we shall assume that $\widehat{f}\,'(t)$ is concave. Suppose that $a > 0$. For $-a < r < a$ $v_a(r)$ is concave by Theorem 3.2 hence is decreasing if $r > 0$ due to symmetry. For $r > a$ we apply Lemma 1.1 and obtain

$$v_a(r) = \left[\frac{\mathcal{H}_f(r+a, r-a; x, y)}{\sqrt{xy}}\right]^{a/r}.$$

Taking the logarithm we get

$$\log v_a(r) = a\frac{\log S_{2a}(r-a) - \log S_{2a}(-a)}{r},$$

where S is defined in Theorem 3.6. $\log S_{2a}(t)$ is concave, so its divided difference decreases. □

4. Comparison of $\mathcal{H}_f$ and $\mathcal{H}_g$

It is natural to ask whether $\mathcal{H}_f$ and $\mathcal{H}_g$ can be compared. The identity (1.11) shows that the inequality $\mathcal{H}_f \leq \mathcal{H}_g$ reverses when p, q change signs. The next theorem establises sufficient and necessary conditions for the ineqality to hold for $p + q > 0$.

Theorem 4.1. *The conditions are equivalent*

(a) *The inequality*

$$\mathcal{H}_f(p, q; x, y) \leq \mathcal{H}_g(p, q; x, y)$$

holds for all $x, y > 0$ and all $p + q > 0$.

(b) *$(\tilde{f}/\tilde{g})(t)$ increases for $0 < t \leq 1$.*

(c) *$(\tilde{f}/\tilde{g})(t)$ decreases for $t > 1$.*

(d) *$\widehat{f}(t) - \widehat{g}(t)$ increases for $t < 0$.*

(e) *$\widehat{f}(t) - \widehat{g}(t)$ decreases for $t < 0$.*

Proof. The equivalence (b)⇔(c) follows from (1.2). Obviously (b) and (d) are equivalent and so are (c) and (d). For $0 < p < q$ and $y = 1$ the inequality (a) is equivalent to $(\tilde{f}/\tilde{g})(x^q) \leq (\tilde{f}/\tilde{g})(x^p)$, which shows that (a) implies (b) and (c). This also shows that (b) and (c) imply (a) in case of positive parameters p, q. To complete the proof we apply the Lemma 1.1 and obtain

$$\left[\frac{\mathcal{H}_f(p, q; x, y)}{\mathcal{H}_g(p, q; x, y)}\right]^{\frac{1}{p+q}} = \left[\frac{\mathcal{H}_f(|p|, |q|; x, y)}{\mathcal{H}_g(|p|, |q|; x, y)}\right]^{\frac{1}{|p|+|q|}},$$

hence the inequality (a) holds for $p + q > 0$. □

Note: the condition (c) is denoted in [4] by $\tilde{f} \preceq \tilde{g}$ and called strong inequality, so our theorem can be restated as follows

Theorem 4.2. *The inequality*

$$\mathcal{H}_f(p, q; x, y) \leq \mathcal{H}_g(p, q; x, y)$$

holds for all $x, y > 0$ and all $p + q > 0$ if and only if $\tilde{f} \preceq \tilde{g}$.

For real α the function $f_\alpha(x, y) = f(x^\alpha, y^\alpha)^{1/\alpha}$ generates $\mathcal{H}_{f_\alpha}(p, q; x, y) = \mathcal{H}_f(\alpha p, \alpha q; x, y)$ so the Corollary 3.5 yields

Corollary 4.3. *If $\widehat{f}'(t)$ is concave for $t \geq 0$ and $\widehat{f}(t)$ is increasing then for $\alpha < \beta$ the strong inequality $\tilde{f}_\alpha \preceq \tilde{f}_\beta$ holds.*

5. Four-parameter family

If f is positively homogeneous then so are $\mathcal{H}_f$ for every (r,s) and we can create a four-parameter family in the same way:

$$\mathcal{F}_f(p,q;r,s;x,y) = \mathcal{H}_{\mathcal{H}_f(r,s)}(p,q;x,y). \tag{5.1}$$

Now we can easily apply the results from previous chapters, because we have simple formula

$$\widehat{\mathcal{H}_f(r,s)}(t) = \frac{\widehat{f}(rt) - \widehat{f}(st)}{r-s} \tag{5.2}$$

Theorem 5.1. *All members of the four-parameter family are means if and only if $t\widehat{f}'(t)$ is increasing.*

Proof. By Theorem 2.1 all $\mathcal{F}_f$ are means if and only if all $\mathcal{H}_f$ increase in x and y, and this is equivalent to monotonicity of $t\widehat{f}'(t)$ by Theorem 2.3. □

Theorem 5.2. *$\mathcal{F}_f$ increases (decreases) in p and q if and only if $r+s>0$ and $t^2\widehat{f}''(t)$ increases (decreases) for $t>0$ or $r+s<0$ and $t^2\widehat{f}''(t)$ decreases (increases) for $t>0$.*

Proof. By 1.6 the function $t^2\widehat{f}''(t)$ is even. Applying Theorem 2.2 it is enough to check convexity of $\widehat{\mathcal{H}_f(r,s)}(t)$.

$$\widehat{\mathcal{H}_f(r,s)}''(t) = \frac{r^2\widehat{f}''(rt) - s^2\widehat{f}''(st)}{r-s} = \frac{1}{t}\frac{r^2t^2\widehat{f}''(rt) - s^2t^2\widehat{f}''(st)}{rt-st} \tag{5.3}$$

and by Lemma 3.3 the convexity depends on monotonicity of $t^2\widehat{f}''(t)$ and the sign of $\frac{st+rt}{t} = s+r$. □

Theorem 5.3. *If $r+s>0$ the following conditions are equivalent:*

(a) *For all $p,q \geq 0$ and all $x,y>0$ $\log\mathcal{F}_f$ is convex (concave) in p and q.*
(b) *For all $p,q \geq 0$ and all $x,y>0$ $\log\mathcal{F}_f$ is Schur-convex (Schur-concave) in p and q.*
(c) *$t^3\widehat{f}'''(t)$ increases (decreases) for $t \geq 0$.*
(d) *For all $p,q \leq 0$ and all $x,y>0$ $\log\mathcal{F}_f$ is concave (convex) in p and q.*
(e) *For all $p,q \leq 0$ and all $x,y>0$ $\log\mathcal{F}_f$ is Schur-concave (Schur-convex) in p and q.*
(f) *$t^3\widehat{f}'''(t)$ decreases (increases) for $t \leq 0$.*

If $r+s<0$ then the conditions (c) and (f) reverse.

Proof. Assume $r+s>0$ and $t>0$. By Theorem 3.2 it is enough to check convexity of $\widehat{\mathcal{H}_f(r,s)}'$. We have

$$\widehat{\mathcal{H}_f(r,s)}'''(t) = \frac{r^3\widehat{f}'''(rt) - s^3\widehat{f}'''(st)}{r-s} = \frac{1}{t^2}\frac{r^3t^3\widehat{f}'''(rt) - s^3t^3\widehat{f}'''(st)}{rt-st}$$

and again the theorem follows from Lemma 3.3. □

Theorem 5.4. *The four-parameter means $\mathcal{F}_f$ increase in x,y if and only if $t\left[t\widehat{f}'\right]'$ increases.*

Proof. By Theorem 2.3 $\mathcal{F}_f$ increases in x, y if and only if $t\widehat{\mathcal{H}_f(r,s)}'(t)$ increase. Differentiating we get

$$\left[t\widehat{\mathcal{H}_f(r,s)}'(t)\right]' = \frac{r\widehat{f}\,'(rt) + r^2 t\widehat{f}\,''(rt) - s\widehat{f}\,'(st) + s^2 t\widehat{f}\,''(st)}{r-s}$$

and the theorem follows. □

Till the end of this section we shall assume that f generates four-parameter family of means. Let us have a closer look at convexity of S-means defined in Theorem 3.6. In our case

$$S_1(t;r,s;x,y) = \mathcal{F}_f(t+1,t;r,s;x,y) = \frac{\mathcal{H}_f(r,s;x^{t+1},y^{t+1})}{\mathcal{H}_f(r,s;x^t,y^t)}$$

From Theorems 3.6 and 5.3 we know that if $r+s>0$ and $t^3\widehat{f}\,'''(t)$ decreases (increases) for $t>0$ then the function $S_1(t)$ is log-concave (log-convex) for $t>-1/2$ and log-convex (log-concave) otherwise. In this section we investigate the function

$$V(t) = \log S_1\left(t, \frac{r}{2t+1}, \frac{s}{2t+1}; x, y\right).$$

A simple calculation shows that the function V is symmetric with respect to the line $t=-1/2$.
The main result we aim to prove here is the following

Theorem 5.5. *If $\widehat{\mathcal{H}_f(r,s)}\,'(t)$ is concave (convex) for $t>0$ then $V(t)$ increases (decreases) and is concave (convex) for $t>-1/2$.*

For $t>-1/2$ let $\overline{t} = \frac{-t}{2t+1}$. The function $t \to \overline{t}$ maps the half-line $(-1/2,\infty)$ onto itself, is decreasing, $\overline{\overline{t}} = t$ and $(2t+1)(2\overline{t}+1)=1$.
The function S_1 satisfies the identity

$$S_1(t;r,s;x,y) = (xy)^{-t} S_1^{2t+1}\left(\tfrac{-t}{2t+1}; (2t+1)r, (2t+1)s\right). \tag{5.4}$$

To show it, let $\mu = 2t+1$ and $\nu = -t/(2t+1)$. Then $-t=\mu\nu$, and $t+1=\mu(\nu+1)$. Using identities (1.8), (1.9), (1.10), we obtain

$$\begin{aligned} S_1(t;r,s;x,y) &= \frac{\mathcal{H}_f(r,s;x^{t+1},y^{t+1})}{\mathcal{H}_f(r,s;x^t,y^t)} = \frac{\mathcal{H}_f(r,s;x^{t+1},y^{t+1})}{(xy)^t\mathcal{H}_f(r,s;x^{-t},y^{-t})} \\ &= (xy)^{-t}\frac{\mathcal{H}_f(r,s;x^{\mu(\nu+1)},y^{\mu(\nu+1)})}{\mathcal{H}_f(r,s;x^{\mu\nu},y^{\mu\nu})} \\ &= (xy)^{-t} S_1(\nu;r,s;x^\mu,y^\mu) \\ &= (xy)^{-t} S_1^{2t+1}\left(\tfrac{-t}{(2t+1)}; (2t+1)r, (2t+1)r; x, y\right). \end{aligned}$$

The identity (5.4) can be written in the form

$$(xy)^t S_1(t;r,s) = S_1^{2t+1}\left(\overline{t}; \frac{r}{2\overline{t}+1}, \frac{s}{2\overline{t}+1}\right). \tag{5.5}$$

Now we can prove Theorem 5.5:

Proof. Let $-1/2 < u < v$. Then $-1/2 < \overline{v} < \overline{u}$ and we can write $\overline{v}$ as a convex combination of $-1/2$ and $\overline{u}$

$$\overline{v} = -\tfrac{1}{2}\tfrac{2\overline{u}-2\overline{v}}{2\overline{u}+1} + \tfrac{2\overline{v}+1}{2\overline{u}+1}\overline{u}.$$

The log-concavity of S_1 implies the inequality

$$S_1^{\frac{2\overline{u}-2\overline{v}}{2\overline{u}+1}}(-1/2;r,s)S_1^{\frac{2\overline{v}+1}{2\overline{u}+1}}(\overline{u};r,s)\le S_1(\overline{v};r,s)$$

and since $S_1(-1/2;r,s,x,y)=\sqrt{xy}$ we have

$$(xy)^{\frac{\overline{u}}{2\overline{u}+1}}S_1^{\frac{1}{2\overline{u}+1}}(\overline{u};r,s)\le (xy)^{\frac{\overline{v}}{2\overline{v}+1}}S_1^{\frac{1}{2\overline{v}+1}}(\overline{v};r,s)$$

and applying (5.5) we obtain

$$S_1\left(u;\frac{r}{(2u+1)},\frac{s}{(2u+1)}\right)\le S_1\left(v;\frac{r}{(2v+1)},\frac{s}{(2v+1)}\right). \tag{5.6}$$

so the monotonicity is proved (obviously if S_1 is log-convex the inequalities are reversed). To show thet V is concave it is enough to prove that for fixed $v>-1/2$ the function

$$m(u)=\frac{V(u)-V(v)}{u-v}$$

is decreasing. Let

$$n(u)=\frac{\log S_1(u;r,s;x,y)-\log S_1(v;r,s;x,y)}{u-v}.$$

Since $\log S_1$ is concave n is decreasing and applying once more (5.5) we have

$$\begin{aligned}
n(\overline{u})&=-\log(xy)+\frac{1}{v-u}\left[\frac{V(u)}{(2\overline{v}+1)}-\frac{V(v)}{(2\overline{u}+1)}\right]\\
&=-\log(xy)+\frac{(2v+1)V(u)-(2u+1)V(v)}{v-u}\\
&=-\log(xy)+2V(v)-(2v+1)\frac{V(v)-V(u)}{v-u}\\
&=-\log(xy)+2V(v)-(2v+1)m(u)
\end{aligned}$$

This means that n and m are of the same monotonicity and the proof is complete. □

6. Applications

6.1. **Geometric mean.** One can easily check that if $f(x,y)=\sqrt{xy}=G(x,y)$ then for every p,q $\mathcal{H}_f(p,q;x,y)=\sqrt{xy}$.

6.2. **Arithmetic mean.** Taking $f(x,y)=A(x,y)=\frac{x+y}{2}$ we obtain Gini means

$$\operatorname{Gini}(p,q;x,y)=\begin{cases}\left(\dfrac{x^q+y^q}{x^p+y^p}\right)^{1/(q-p)} & q\ne p,\\ \exp\left(\dfrac{x^p\log x+y^p\log y}{x^p+y^p}\right) & q=r.\end{cases} \tag{6.1}$$

We have

$$\widehat{A}(2t)=\log\frac{e^{2t}+1}{2}=t+\log\cosh t \tag{6.2}$$

$$2\widehat{A}\,'(2t)=1+\tanh t>0 \tag{6.3}$$

$$4\widehat{A}\,''(2t)=\frac{1}{\cosh^2 t}>0 \tag{6.4}$$

$$8\widehat{A}\,'''(2t)=-2\frac{\sinh t}{\cosh^3 t} \tag{6.5}$$

so by (6.3) and Theorem 2.1

Property of Gini means 1. For every p, q $\operatorname{Gini}(p,q;x,y)$ are means.

Combining (6.4) and Theorem 2.3 we see that

Property of Gini means 2. $\operatorname{Gini}(p,q;x,y)$ increases in p and q.

By (6.5) $\widehat{A}'(t)$ is concave for $t>0$ and convex for $t<0$ so Theorem 3.2 yields

Property of Gini means 3. $\operatorname{Gini}(p,q)$ is logarithmically concave in $pq>0$ and logarithmically convex for $pq<0$.

As $A(0,1)=1/2$, Theorem 2.5 implies that Gini means are not monotone in x, y for $p, q>0$. However, Corollary 2.4 shows that they are monotone if $pq<0$.

The following result of Horst Alzer ([2]) is a consequence of Theorem 3.6:

Corollary 6.1. *For fixed x, y*

$$K(r;x,y)=\operatorname{Gini}(r+1,r;x,y)=\frac{x^{r+1}+y^{r+1}}{x^r+y^r}$$

is increasing and log-concave for $r>-1/2$, and log-convex otherwise.

6.3. **Logarithmic mean.** The logarithmic mean $f(x,y)=L(x,y)=\frac{x-y}{\log x-\log y}$ leads to Stolarsky means

$$E(p,q;x,y)=\begin{cases}\left(\dfrac{p}{q}\dfrac{y^q-x^q}{y^p-x^p}\right)^{1/(q-p)} & qp(q-p)(x-y)\neq 0,\\ \left(\dfrac{1}{p}\dfrac{y^p-x^p}{\log y-\log x}\right)^{1/p} & p(x-y)\neq 0,\ q=0,\\ e^{-1/p}\left(y^{y^p}/x^{x^p}\right)^{1/(y^p-x^p)} & p=q,\ p(x-y)\neq 0,\\ \sqrt{xy} & p=q=0,\\ x & x=y.\end{cases} \tag{6.6}$$

In this case

$$\widehat{L}(2t)=\log\frac{e^{2t}-1}{2t}=t+\log\frac{\sinh t}{t} \tag{6.7}$$

$$2\widehat{L}'(2t)=2\left(\frac{1}{1-e^{-2t}}-\frac{1}{2t}\right)=1+\frac{\cosh t}{\sinh t}-\frac{1}{t}>0 \tag{6.8}$$

$$4\widehat{L}''(2t)=\frac{1}{t^2}-\frac{1}{\sinh^2 t}>0 \tag{6.9}$$

$$8\widehat{L}'''(2t)=-2\frac{\sinh^3 t-t^3\cosh t}{t^3\sinh^3 t} \tag{6.10}$$

$\widehat{L}$ increases, so by Theorem 2.1

Property of Stolarsky means 1. For every p, q $E(p,q;x;y)$ is a mean.

Theorem 2.2 combined with (6.9) shows

Property of Stolarsky means 2. E is increasing in p and q.

The function $\frac{e^{-t}-1}{t}$ is increasing as the divided difference of the convex function e^{-t}, hence so is $t\widehat{L}'(t)=\frac{t}{1-e^{-t}}-1$, therefore

Property of Stolarsky means 3. E increases in x and y.

To investigate further properties of Stolarsky means we need the following

Lemma 6.2. *The function*

$$h(t) = \frac{t^3 \cosh t}{\sinh^3 t}$$

increases from 0 *to* 1 *on* $(-\infty, 0)$ *and decreases on* $(0, \infty)$.

Proof. It is clear that $h(0) = 1$ and $h(\pm\infty) = 0$, and since it is even all we have to do is to show that it decreases for positive t. Direct diffrentiation leads to quite complicated inequality, so let us make a little trick here: let

$$g(t) = \frac{\sinh t}{\cosh^{1/3} t}.$$

then

$$\begin{aligned} g'(t) =& \frac{2}{3}\cosh^{2/3} t + \frac{1}{3}\cosh^{-4/3} t, \\ g''(t) =& \frac{4}{9}\sinh t \cosh^{-1/3} t \left(1 - \cosh^{-2} t\right), \end{aligned}$$

so g is convex for $t \geq 0$, therefore its divided difference $g(t)/t$ increases and h is its cubed reciprocal. □

From this Lemma and (6.10) we see that $\widehat{L}\,'(t)$ is concave for $t > 0$ and convex otherwise, so by Theorem 3.2

Property of Stolarsky means 4. E is logarithmically concave in variables p, q in the quadrant $p, q > 0$ and logarithmically convex in $p, q < 0$

The following result of Horst Alzer ([1]) is a consequence of Theorem 3.6:

Corollary 6.3. *For fixed* x, y

$$J(r; x, y) = E(r + 1, r; x, y) = \frac{r}{r+1} \frac{x^{r+1} - y^{r+1}}{x^r - y^r}$$

is increasing and log-concave for $r > -1/2$*, and log-convex otherwise.*

Consider now some one-parameter families generated by classical means:

- power means

$$M(r; x, y) = \left(\frac{x^r + y^r}{2}\right)^{1/r} = E(r, 2r; x, y),$$

- Heronian means

$$H(r; x, y) = \left(\frac{x^r + \sqrt{xy}^r + y^r}{3}\right)^{1/r} = E(3r/2, r/2; x, y),$$

- Identric means

$$I(r; x, y) = e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r - x^r)} = E(r, r; x, y),$$

- Stolarsky means

$$L(r; x, y) = \left(\frac{1}{r}\frac{x^r - y^r}{\log x - \log y}\right)^{1/r} = E(r, 0; x, y).$$

They are all monotone in r and by Theorem 3.4 they are log-concave for $r > 0$ and log-convex otherwise.

Classical result of Tung-Po Lin ([5]) states that $L(x,y) \leq M(1/3;x,y)$. Let us refine it:

Corollary 6.4.

$$L(x,y) \leq L^{1/3}(x,y)I^{2/3}(1/2;x,y) \leq M(1/3;x,y)$$

Proof. We have

$$L = E(0,1), \quad M(1/3) = E(1/3,2/3), \quad I(1/2) = E(1/2,1/2).$$

As $E(p,q)$ is log-concave in $pq > 0$ we have $M(1/3) \geq L^{1/3}I^{2/3}(1/2)$, and Theorem 3.8 implies that $L \leq M(1/3) \leq I(1/2)$ □

6.4. **Logarithmic mean once more.** Consider now the four-parameter means generated by the logaritmic mean (in other words the two-parameter means generated by the Stolarsky means $E(r,s;x,y)$). They are important, because they contain two-parameter families generated by logarithmic, Heronian, arithmetic and centroidal means :

$$\mathcal{F}_L(0,1;r,s;x,y) = E(r,s;x,y),$$

$$\mathcal{F}_L(1/2,3/2;r,s;x,y) = N(r,s;x,y) = \left(\frac{x^s + (\sqrt{xy})^s + y^s}{x^r + (\sqrt{xy})^r + y^r}\right)^{1/s-r},$$

$$\mathcal{F}_L(1,2;r,s;x,y) = \mathrm{Gini}(r,s;x,y),$$

$$\mathcal{F}_L(0,1;r,s;x,y) = T(r,s;x,y) = \left(\frac{x^{2s} + (xy)^s + y^{2s}}{x^s + y^s} \Big/ \frac{x^{2r} + (xy)^r + y^{2r}}{x^r + y^r}\right)^{1/s-r}.$$

Stolarsky means increase in x and y, thus $\mathcal{F}_L$ are means, but in general they are not monotone (Gini means are not monotone).

Formula (6.9) combined with Theorem 5.2 shows that $\mathcal{F}_l(p,q;r,s)$ increase in p,q if $r+s > 0$

Lemma 6.2 and (6.10) show that $t^3\widehat{L}'''(t)$ decreases for $t > 0$, thus by Theorem 5.3 $\mathcal{F}_L$ is log-concave in the quadrant $p,q > 0$ if $r + S > 0$. Additionally by Theorem 3.6 the function

$$\mathcal{F}_L(t+1,t;r,s;x,y) = S(t;r,s;x,y) = \frac{E(r,s;x^{t+1},y^{t+1})}{E(r,s;x^t,y^t)}$$

is log-concave in t for $t > -1/2$ and log-convex otherwise and in consequence increasing in t. This establishes inequalities

$$E(r,s) < N(r,s) < \mathrm{Gini}(r,s) < T(r,s)$$

valid for $r+s > 0$ and $x \neq y$. Theorem 5.5 implies also stronger inequalities (see [14] and references therein:

$$E(r,s) < N(r.2,s/2) < \mathrm{Gini}(r/3,s/3) < T(r/5,s/5).$$

6.5. **Product function.** If $f_1,\ldots,f_n : \mathbf{R}_+^2 \to \mathbf{R}_+$ are positively homogeneous and symmetric, satisfy $f_i(1,1,) = 1$, $\alpha_1,\ldots,\alpha_n$ are positive and $\alpha_1 + \cdots + \alpha_n = 1$, then

$$f(x,y) = \prod_{i=1}^{n} f_i\left(x^{\alpha_i}, y^{\alpha_i}\right)$$

has the same property. In particular if all f_i's are means then f is also a mean. Clearly

$$\widehat{f}(t) = \sum_{i=1}^{n} \widehat{f}_i(\alpha_i t),$$

so if f_i's generate means, monotone or log-convex two-parameter families, then so does their product.
We give two examples here:

6.5.1. *Heinz means.* For $0 \le \alpha \le 1/2$ Heinz means are defined by

$$A_\alpha(x,y) = \frac{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha}{2} = G\left(x^{2\alpha}, y^{2\alpha}\right) A\left(x^{1-2\alpha}, y^{1-2\alpha}\right).$$

Both G and A generate two-parameter means that are increasing in p and q, log-concave in p, q for $p, q > 0$, so the two-parameter means generated by A_α have the same properties. Monotonicity in x, y is interesting, because Heinz means establish homotopy between monotone ($\alpha = 1/2$) and nonmonotone ($\alpha = 0$) families. Let us check when they fail to be monotone. By Theorem 2.3 we have to investigate when $t\widehat{A_\alpha}'(t)$ is increasing:

$$\left[t\widehat{A_\alpha}'(t)\right]' = \frac{1}{2} + \left(\frac{1}{2} - \alpha\right) u\left(\left(\frac{1}{2} - \alpha\right) t\right),$$

where $u(t) = \frac{\sinh t \cosh t + t}{\cosh^2 t}$. The function u attains its minimun $M \approx -1.999679$ at $t \approx -1.1995$, so the two-parameter families generated by A_α are increasing in x and y for $\alpha \ge 0.24996....$

6.5.2. *logarithmic analogue of Heinz means.* Using the logaritmic instead of the arithmetic mean we get

$$\begin{aligned} L_\alpha(x,y) =& \frac{x^\alpha y^{1-\alpha} - x^{1-\alpha} y^\alpha}{(1-2\alpha)(\log y - \log x)} \\ =& G\left(x^{2\alpha}, y^{2\alpha}\right) L\left(x^{1-2\alpha}, y^{1-2\alpha}\right) = \frac{1}{1-2\alpha} \int_\alpha^{1-\alpha} x^s y^{1-s} ds. \end{aligned}$$

Obviously, the two-parameter families admit the same properties as Stolarsky means.

6.6. **Seiffert mean.** The Seiffert mean

$$P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}}$$

was introduced in [6]. Peter Hästö proved in [4] that $M(1/2) \preceq P \preceq M(2/3)$ and that the constants $1/2$ and $2/3$ cannot be improved, therefore, by Theorem 4.1 the inequalities

$$\text{Gini}(p/2, q/2; x, y) \le \mathcal{H}_P(p, q; x, y) \le \text{Gini}(2p/3; 2q/3; x, y)$$

hold for all p, q such that $p + q > 0$.

6.7. **Nondiffrentiable case.** Consider now two-parameter means generated by $\max(x,y)$ and $\min(x,y)$ (means, because max and min are monotone in both variables). We have

$$\widehat{\max}(t) = \max(t, 0), \quad \widehat{\min}(t) = \min(t, 0)$$

Elementary calculations lead to the following formulae:

$$\mathcal{H}_{\max}(p, q; x, y) = \sqrt{xy}\sqrt{\frac{\max(x,y)}{\min(x,y)}}^{\frac{p+q}{|p|+|q|}}$$

and

$$\mathcal{H}_{\min}(p,q;x,y) = \sqrt{xy}\sqrt{\frac{\min(x,y)}{\max(x,y)}}^{\frac{p+q}{|p|+|q|}}$$

Applying our results we see that $\mathcal{H}_{\max}$ increases and $\mathcal{H}_{\min}$ decreases in p,q, both increase in x,y ($\widehat{\max}$ is not differentiable, but $t\widehat{\max}'(t)$ can be interpreted as $[t\widehat{\max}(t)]'(t) - \widehat{\max}(t)$). Application of Theorem 3.2 is immaterial here, because in areas $pq > 0$ our functions are constant in variables p,q, but Theorem 4.1 gives an interesting (in case pq$<$0) result:

Corollary 6.5. *If for some f and all p,q $\mathcal{H}_f(p,q)$ are means, then for $p+q>0$*

$$\mathcal{H}_{\min}(p,q;x,y) \le \mathcal{H}_f(p,q;x,y) \le \mathcal{H}_{\max}(p,q;x,y).$$

Proof. By Theorem 2.1 $\tilde{f}$ increases, $\widetilde{\max}$ is constant in $(0,1)$ and $\widetilde{\min}$ is constatnt in $(1,\infty)$ thus by Theorem 4.1 required inequalities are valid. □

6.8. **Application of comparison result.** For $w \ge 0$ the weighted Heronian mean is defined by

$$h_w(x,y) = \frac{x + w\sqrt{xy} + y}{2+w}.$$

Clearly $h_0 = A$ and $h_\infty = G$. If $w > v \ge 0$, then

$$\frac{\widetilde{h_w}(t)}{\widetilde{h_v}(t)} = \frac{2+v}{2+w}\left(1 + \frac{w-v}{\sqrt{t} + v + 1/\sqrt{t}}\right)$$

increases for $0 < t < 1$, so $h_w \preceq h_v$. and by Theorem 4.1 we have

$$\mathcal{H}_{h_w}(p,q;x,y) \le \mathcal{H}_{h_v}(p,q;x,y)$$

for all $x,y > 0$ and p,q such that $p+q>0$.

7. Open questions

If f is homogeneous of order 1, then so are $\mathcal{H}_f(p,q)$ for every p,q. We can iterate this process bnulding a sequence of $2n$-parameter functions $\mathcal{H}_{f,n}$. The geometric mean is a fixed point of this operation. Examples above show, that means do not necessary generate means. It would be interesting to answer the following questions:

- Does there exist for every n a function f such that for all $k \le n$ $\mathcal{H}_{f,k}$ are means. If yes, do they converge in some sense to G?
- Does there exist a function $f \ne G$ such that all $\mathcal{H}_{f,n}$ are means?

It seems that $G(x^\alpha, y^\alpha)L(x^{1-\alpha}, y^{1-\alpha})$ for α sufficiently close to 1 can give positive answer to the first question.

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