A NEW PROOF METHOD OF ANALYTIC INEQUALITY

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Abstract. This paper gives a new proof method of analytic inequality involving n variables. As its Applications, we proved some well-known inequalities and improved the Carleman-Inequality.

1. MONOTONICITY ON SPECIAL REGION

Throughout the paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ denotes the set of strictly positive real numbers, $n \in \mathbb{N}$, $n \geq 2$.

In this section, we shall provide a new proof method of analytic inequality involving n variables.

Theorem 1.1. *Given $a, b \in \mathbb{R}$, $c \in [a,b]$. Let $f : \boldsymbol{x} \in [a,b]^n \to \mathbb{R}$ have continuous partial derivative,*

$$D_i = \left\{ (x_1, x_2, \cdots, x_{n-1}, c) \,|\, \min_{1 \leq k \leq n-1} \{x_k\} \geq c, x_i = \max_{1 \leq k \leq n-1} \{x_k\} \neq c \right\}, i = 1, 2, \cdots, n-1.$$

If $\partial f(\boldsymbol{x}) / \partial x_i > 0$ hold for any $\boldsymbol{x} \in D_i \, (i = 1, 2, \cdots, n-1)$, then

$$f(y_1, y_2, \cdots, y_{n-1}, c) \geq f(c, c, \cdots c, c)$$

hold for $y_i \in [c, b] \, (i = 1, 2, \cdots, n-1)$.

Proof. Without the losing of generality, we let $n = 3$ and $y_1 > y_2 > c$.

For $x_1 \in [y_2, y_1]$, it has $(x_1, y_2, c) \in D_1$, then $\partial f(\boldsymbol{x}) / \partial x_1 |_{\boldsymbol{x} = (x_1, y_2, c)} > 0$. Owing to the continuity of partial derivative and $\partial f(\boldsymbol{x}) / \partial x_1 |_{\boldsymbol{x} = (y_2, y_2, c)} > 0$, it exists ε, such that $y_2 - \varepsilon \geq c$ and $\partial f(\boldsymbol{x}) / \partial x_1 |_{\boldsymbol{x} = (x_1, y_2, c)} > 0$ for any $x_1 \in [y_2 - \varepsilon, y_2]$. Hence, $f(\cdot, y_2, c) : x_1 \in [y_2 - \varepsilon, y_1] \to f(x_1, y_2, c)$ is strictly monotone increasing,

$$f(y_1, y_2, c) > f(y_2, y_2, c) > f(y_2 - \varepsilon, y_2, c).$$

For $x_2 \in [y_2 - \varepsilon, y_2]$, $(y_2 - \varepsilon, x_2, c) \in D_2$, $\partial f(\boldsymbol{x}) / \partial x_2 |_{\boldsymbol{x} = (y_2 - \varepsilon, x_2, c)} > 0$. Then

$$f(y_1, y_2, c) > f(y_2, y_2, c) > f(y_2 - \varepsilon, y_2, c) > f(y_2 - \varepsilon, y_2 - \varepsilon, c).$$

If $y_2 - \varepsilon = c$, this completes the proof of the Theorem 1.1. Otherwise, we repeat the above process. It is clear that the first variable and the second variable of function f are decreasing and no less than c. Let s, t are their limits, then $f(y_1, y_2, c) > f(s, t, c)$, where $s, t \geq c$. If $s = c, t = c$, this completes the proof of the Theorem 1.1. Otherwise, we repeat the above process. Let the greatest lower bound of the first variable and the second variable are p, q. It is easy to see $p = q = c$, and $f(y_1, y_2, c) > f(c, c, c)$. □

Similarly to the above ,we know Theorem 1.2 is true.

Theorem 1.2. *Given $a, b \in \mathbb{R}$, $c \in [a,b]$. Let $f : \boldsymbol{x} \in [a,b]^n \to \mathbb{R}$ have continuous partial derivative,*

$$D_i = \left\{ (x_1, x_2, \cdots, x_{n-1}, c) \,|\, \max_{1 \leq k \leq n-1} \{x_k\} \leq c, x_i = \min_{1 \leq k \leq n-1} \{x_k\} \neq c \right\}, i = 1, 2, \cdots, n-1.$$

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If $\partial f(\boldsymbol{x})/\partial x_i < 0$ *hold for any* $\boldsymbol{x} \in D_i\,(i = 1, 2, \cdots, n-1)$*, then*

$$f(y_1, y_2, \cdots, y_{n-1}, c) \geq f(c, c, \cdots c, c)$$

hold for $y_i \in [a, c]\,(i = 1, 2, \cdots, n-1)$*.*

In particular, according to Theorem 1.1 and Theorem 1.2, the following four corollaries hold.

Corollary 1.1. *Let* $a, b \in \mathbb{R}$*,* $f : [a, b]^n \to \mathbb{R}$ *have continuous partial derivative,*

$$D_i = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \,| a \leq \min_{1\leq k\leq n} \{x_k\} < x_i = \max_{1\leq k\leq n} \{x_k\} \leq b \right\}, \quad i = 1, 2, \cdots, n$$

If $\partial f(\boldsymbol{x})/\partial x_i > 0$ *hold for any* $\boldsymbol{x} \in D_i$ *and any* $i = 1, 2, \cdots, n$*, then*

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min})$$

hold for $x_i \in [a, b]\,(i = 1, 2, \cdots, n)$*, with* $x_{\min} = \min\limits_{1\leq k\leq n} \{x_k\}$*.*

Corollary 1.2. *Supposes* $a, b \in \mathbb{R}$*,*

$$D_1 = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \,| a \leq \min_{1\leq k\leq n} \{x_k\} < x_1 = \max_{1\leq k\leq n} \{x_k\} \leq b \right\}.$$

Let $f : [a, b]^n \to \mathbb{R}$ *be symmetric, all partial differentiations of* f *be continuous. If* $\partial f(\boldsymbol{x})/\partial x_1 > 0$ *hold for* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in D_1$*, then*

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\min}, x_{\min}, \cdots, x_{\min}),$$

with $x_{\min} = \min\limits_{1\leq k\leq n} \{x_k\}$*. Equality holds if and only if* $x_1 = x_2 = \cdots = x_n$*.*

Corollary 1.3. *Supposes* $a, b \in \mathbb{R}$*,* $f : [a, b]^n \to \mathbb{R}$ *have continuous partial derivative,*

$$D_i = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \,| a \leq x_i = \min_{1\leq k\leq n} \{x_k\} < \max_{1\leq k\leq n} \{x_k\} \leq b \right\}.$$

If $\partial f(\boldsymbol{x})/\partial x_i < 0$ *hold for any* $\boldsymbol{x} \in D_i$ *and any* $i = 1, 2, \cdots, n$*, then*

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\max}, x_{\max}, \cdots, x_{\max}),$$

with $x_{\max} = \max\limits_{1\leq k\leq n} \{x_k\}$*. Equality holds if and only if* $x_1 = x_2 = \cdots = x_n$*.*

Corollary 1.4. *Supposes* $a, b \in \mathbb{R}$*,*

$$D_n = \left\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) \,| a \leq x_n = \min_{1\leq k\leq n} \{x_k\} < \max_{1\leq k\leq n} \{x_k\} \leq b \right\}.$$

Let $f : [a, b]^n \to \mathbb{R}$ *be symmetric, all partial differentiations of* f *be continuous. If* $\partial f(\boldsymbol{x})/\partial x_n < 0$ *hold for* $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in D_n$*, then*

$$f(x_1, x_2, \cdots, x_n) \geq f(x_{\max}, x_{\max}, \cdots, x_{\max}),$$

with $x_{\max} = \max\limits_{1\leq k\leq n} \{x_k\}$*. Equality holds if and only if* $x_1 = x_2 = \cdots = x_n$*.*

2. Unifying Proof of Some Well-known Inequality

In this section, we denote $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$, $a_{\min} = \min\limits_{1 \le k \le n} \{a_k\}$, $a_{\max} = \max\limits_{1 \le k \le n} \{a_k\}$ and

$$D_i = \{\boldsymbol{a} | a_i = a_{\max} > a_{\min} > 0\}, \quad i = 1, 2, \cdots, n,$$

Proposition 2.1. *(Power Mean Inequality) The power mean $M_r(\boldsymbol{a})$ of order r with respect to the positive numbers $a_1, a_2, \cdots, a_n$ is defined as $M_r(\boldsymbol{a}) = \left(\frac{1}{n}\sum_{i=1}^n a_i^r\right)^{\frac{1}{r}}$ for $r \neq 0$, and $M_0(\boldsymbol{a}) = \prod_{i=1}^n a_i^{\frac{1}{n}}$. Then $M_r(\boldsymbol{a}) \ge M_s(\boldsymbol{a})$ if $r > s$, equality holds if and only if $a_1 = a_2 = \cdots = a_n$.*

Proof. Obviously, $M_r(\boldsymbol{a})$ is symmetric with respect to $a_1, a_2, \cdots, a_n$, $r \mapsto M_r(\boldsymbol{a})$ is continuous. Without the losing of generality, we let $r, s \neq 0$,

$$f(\boldsymbol{a}) = \frac{1}{r} \ln\left(\frac{\sum_{i=1}^n a_i^r}{n}\right) - \frac{1}{s} \ln\left(\frac{\sum_{i=1}^n a_i^s}{n}\right), \boldsymbol{a} \in \mathbb{R}_+^n.$$

Then

$$\begin{aligned}
\frac{\partial f(\boldsymbol{a})}{\partial a_1} &= \frac{a_1^{r-1}}{\sum_{i=1}^n a_i^r} - \frac{a_1^{s-1}}{\sum_{i=1}^n a_i^s} \\
&= \frac{\sum_{i=2}^n \left(a_1^{r-1} a_i^s - a_1^{s-1} a_i^r\right)}{\sum_{i=1}^n a_i^r \cdot \sum_{i=1}^n a_i^s} \\
&= \frac{\sum_{i=2}^n a_1^{s-1} a_i^r \left[(a_1/a_i)^{r-s} - 1\right]}{\sum_{i=1}^n a_i^r \cdot \sum_{i=1}^n a_i^s}.
\end{aligned}$$

If $\boldsymbol{a} \in D_1$, we get $\partial f(\boldsymbol{a})/\partial a_1 > 0$. According to Corollary 1.2, it has

$$f(a_1, a_2, \cdots, a_n) \ge f(a_{\min}, a_{\min}, \cdots, a_{\min}),$$

$$\left(\frac{\sum_{i=1}^n a_i^r}{n}\right)^{1/r} \ge \left(\frac{\sum_{i=1}^n a_i^s}{n}\right)^{1/s}, \qquad M_r(\boldsymbol{a}) \ge M_s(\boldsymbol{a}).$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$. □

Proposition 2.2. *(Holder-Inequality) Let $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$, $p, q > 1$, and $1/p + 1/q = 1$. Then*

$$\left(\sum_{k=1}^n x_k^p\right)^{1/p} \left(\sum_{k=1}^n y_k^q\right)^{1/q} \ge \sum_{k=1}^n x_k y_k.$$

Proof. Let $\boldsymbol{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}_+^n$,

$$f : \boldsymbol{a} \in \mathbb{R}_+^n \to \left(\sum_{k=1}^n b_k\right)^{1/p} \left(\sum_{k=1}^n b_k a_k\right)^{1/q} - \sum_{k=1}^n b_k a_k^{1/q}, \quad \boldsymbol{a} \in \mathbb{R}_+^n.$$

If $\boldsymbol{a} \in D_1$,

$$\begin{aligned}
\frac{\partial f(\boldsymbol{a})}{\partial a_1} &= \frac{1}{q} b_1 \left(\sum_{k=1}^n b_k\right)^{1/p} \left(\sum_{k=1}^n b_k a_k\right)^{1/q-1} - \frac{1}{q} b_1 a_1^{1/q-1} \\
&= \frac{1}{q} b_1 a_1^{-1/p} \left(\sum_{k=1}^n b_k a_k\right)^{-1/p} \left[\left(\sum_{k=1}^n b_k\right)^{1/p} a_1^{1/p} - \left(\sum_{k=1}^n b_k a_k\right)^{1/p}\right] \\
&> \frac{1}{q} b_1 a_1^{-1/p} \left(\sum_{k=1}^n b_k a_k\right)^{-1/p} \left[\left(\sum_{k=1}^n b_k\right)^{1/p} a_1^{1/p} - \left(\sum_{k=1}^n b_k a_1\right)^{1/p}\right] \\
&= 0.
\end{aligned}$$

Similarly, If $\boldsymbol{a} \in D_i (i = 2, 3, \cdots, n)$, $\partial f(\boldsymbol{a})/\partial a_i > 0$. According to Theorem 1.1,

$$f(a_1, a_2, \cdots, a_n) \ge f(a_{\min}, a_{\min}, \cdots, a_{\min}),$$

$$\left(\sum_{k=1}^{n} b_k\right)^{1/p}\left(\sum_{k=1}^{n} b_k a_k\right)^{1/q} \geq \sum_{k=1}^{n} b_k a_k^{1/q}.$$

In above inequality, let $a_k = y_k^q/x_k^p$, $b_k = x_k^p$, we complete the proof of Proposition 2.2. □

Proposition 2.3. *(Minkowski-Inequality) Let* $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$, $p > 1$, *then*

$$\left(\sum_{k=1}^{n} x_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} y_k^p\right)^{1/p} \geq \left(\sum_{k=1}^{n} (x_k + y_k)^p\right)^{1/p}.$$

Proof. Let $\boldsymbol{b} = (b_1, b_2, \cdots, b_n) \in \mathbb{R}_+^n$,

$$f : \boldsymbol{a} \in \mathbb{R}_+^n \to \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p}, \quad \boldsymbol{a} \in \mathbb{R}_+^n.$$

If $\boldsymbol{a} \in D_1$,

$$\begin{aligned}
\frac{\partial f(\boldsymbol{a})}{\partial a_1} &= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1} - \frac{1}{p} b_1 a_1^{1/p-1}\left(a_1^{1/p}+1\right)^{p-1}\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p-1} \\
&= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1}\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1-1/p} - \left(1 + a_1^{-1/p}\right)^{p-1}\left(\sum_{k=1}^{n} b_k a_k\right)^{1-1/p}\right] \\
&= \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1}\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1-1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+a_k^{1/p}a_1^{-1/p}\right)^p\right)^{1-1/p}\right] \\
&> \frac{1}{p} b_1 \left(\sum_{k=1}^{n} b_k a_k\right)^{1/p-1}\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p-1} \\
&\quad \cdot \left[\left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1-1/p} - \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+a_1^{1/p}a_1^{-1/p}\right)^p\right)^{1-1/p}\right] \\
&= 0.
\end{aligned}$$

Similarly, If $\boldsymbol{a} \in D_i (i = 2, 3, \cdots, n)$, $\partial f(\boldsymbol{a})/\partial a_i > 0$. According to Theorem 1.1,

$$f(a_1, a_2, \cdots, a_n) \geq f(a_{\min}, a_{\min}, \cdots, a_{\min}),$$

$$\left(\sum_{k=1}^{n} b_k a_k\right)^{1/p} \geq \left(\sum_{k=1}^{n} b_k \left(a_k^{1/p}+1\right)^p\right)^{1/p}.$$

In above inequality, let $a_k = y_k^p/x_k^p$, $b_k = x_k^p$, we complete the proof of Proposition 2.3. □

3. A Refinement on the Carleman's Inequality

If $a_n \geq 0\,(n \in \mathbb{N}, n \geq 1)$ with $0 < \sum_{n=1}^{\infty} a_n < \infty$, then the famous Carleman's inequality is

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{1/n} < e\sum_{n=1}^{\infty} a_n, \tag{3.1}$$

where the constant factor is the best possible(see [1]).

Recently, Yang et al. [9] gave a strengthened version of (3.1) as follows.

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{1/n} < e\sum_{n=1}^{\infty}\left(1 - \frac{1}{2n+2}\right) a_n \tag{3.2}$$

Some other strengthened version of (3.1) were given by [1]–[9]. In the section, we shall obtain another refinement on the Carleman's inequality in the form of Corollary 3.2.

Lemma 3.1. *Let $i \in \mathbb{N}$, $i \geq 1$, then*

$$e\left(1-\frac{2}{3i+7}\right)\frac{1}{i} > \sum\nolimits_{k=i}^{\infty}\frac{1}{k\,(k!)^{1/k}}, \tag{3.3}$$

$$e\left(1-\frac{2}{3i+10}\right)\frac{1}{i+1} > \frac{1}{((i+1)!)^{1/(i+1)}}. \tag{3.4}$$

Proof. Let $\psi(i) = e\left(1-\frac{2}{3i+7}\right)\frac{1}{i} - \sum_{k=i}^{\infty}\frac{1}{k(k!)^{1/k}}$, then $\psi(i) > \psi(i+1)$ is equivalent to

$$1-\frac{2i+2}{3i+7}+\frac{2i}{3i+10} > \frac{i+1}{e\,(i!)^{1/i}} \tag{3.5}$$

If $1 \leq i \leq 16$, after brief computation, we know inequality (3.5) hold.
If $i \geq 17$, we get $\sqrt{2\pi i} \geq e^{7/3}$,

$$\sqrt{2\pi i} \geq e^{\left(21i^2+71i+70\right)/\left(9i^2+39i+50\right)}. \tag{3.6}$$

If $x > 0$, it have $e > (1+1/x)^x$. Thus

$$e > \left(1+\frac{21i^2+71i+70}{(9i^2+39i+50)\,i}\right)^{\left(9i^2+39i+50\right)i/\left(21i^2+71i+70\right)}. \tag{3.7}$$

By virtue of (3.6) and (3.7), we get

$$\sqrt{2\pi i} > \left(1+\frac{21i^2+71i+70}{(9i^2+39i+50)\,i}\right)^{i}, \quad (2\pi i)^{1/(2i)} > \frac{(i+1)(3i+7)(3i+10)}{i\,(9i^2+39i+50)},$$

$$\frac{i+5}{3i+7}+\frac{2i}{3i+10} > \frac{i+1}{i\,(2\pi i)^{1/(2i)}}. \tag{3.8}$$

The well-known Stirling-equality is $i! = \sqrt{2\pi i}\,(i/e)^i \exp(\theta_i/12i)$ with $0 < \theta_i < 1$. We have

$$i! > \sqrt{2\pi i}\left(\frac{i}{e}\right)^{i}. \tag{3.9}$$

Owing to inequality (3.8) and (3.9), inequality (3.5) hold.
Hence, $\{\psi(i)\}_{i=1}^{\infty}$ is a strictly decreasing sequence. Because $\lim\limits_{i\to+\infty}\psi(i) = 0$, we have $\psi(i) > 0$. Inequality (3.3) is proved.
Meanwhile,

$$\begin{aligned}
&\sqrt{2\pi(i+1)} > e^{2/3},\\
&\sqrt{2\pi(i+1)} > e^{(2i+2)/(3i+8)},\\
&\sqrt{2\pi(i+1)} > \left(1+\frac{2}{3i+8}\right)^{(3i+8)/2\cdot(2i+2)/(3i+8)},\\
&(2\pi(i+1))^{1/(2i+2)} > \frac{3i+10}{3i+8},
\end{aligned}$$

$$e\left(1-\frac{2}{3i+10}\right)\frac{1}{i+1} > \frac{e}{(i+1)\,(2\pi(i+1))^{1/(2i+2)}}. \tag{3.10}$$

According to $(i+1)! > \sqrt{2\pi(i+1)}\,((i+1)/e)^i$, inequality (3.4) hold. □

Theorem 3.1. *Let* $n \in \mathbb{N}, n \geq 1, a_k > 0\,(k = 1, 2, \cdots, n)$*,* $B_n = \min_{1\leq k\leq n} \{ka_k\}$*, then*

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)a_k-\sum_{k=1}^{n}\left(\prod_{j=1}^{k}a_j\right)^{1/k} \geq B_n\left[e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{(k!)^{1/k}}\right]. \tag{3.11}$$

Proof. Let $b_k = ka_k, k = 1, 2, \cdots, n$, $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$

$$D_i = \left\{\boldsymbol{b}|b_i = \max_{1\leq k\leq n}\{b_k\} > \min_{1\leq k\leq n}\{b_k\} > 0\right\}, \quad i = 1, 2, \cdots, n,$$

and

$$f : \boldsymbol{b} \in \mathbb{R}_+^n \to e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{b_k}{k}-\sum_{k=1}^{n}\left(\frac{1}{k!}\prod_{j=1}^{k}b_j\right)^{1/k}, \quad \boldsymbol{b} \in \mathbb{R}_+^n.$$

Then inequality (3.11) is is equivalent to the following (3.12)

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{b_k}{k}-\sum_{k=1}^{n}\left(\frac{1}{k!}\prod_{j=1}^{k}b_j\right)^{1/k} \geq B_n\left[e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{(k!)^{1/k}}\right], \tag{3.12}$$

and $B_n = \min_{1\leq k\leq n}\{b_k\}$.

If $\boldsymbol{b} \in D_i\ (i = 1, 2, \cdots, n)$,

$$\begin{aligned}\frac{\partial f(\boldsymbol{b})}{\partial b_i} &= e\left(1-\frac{2}{3i+7}\right)\frac{1}{i}-\sum_{k=i}^{n}\frac{1}{kb_i}\left(\frac{1}{k!}\prod_{j=1}^{k}b_j\right)^{1/k}\\ &> e\left(1-\frac{2}{3i+7}\right)\frac{1}{i}-\sum_{k=i}^{n}\frac{1}{k\,(k!)^{1/k}}\\ &> e\left(1-\frac{2}{3i+7}\right)\frac{1}{i}-\sum_{k=i}^{\infty}\frac{1}{k\,(k!)^{1/k}}.\end{aligned}$$

According to inequality (3.3), $\partial f(\boldsymbol{b})/\partial b_i > 0$. In view of Theorem 1.1,

$$f\left(b_1, b_2, \cdots, b_n\right) \geq f\left(B_n, B_n, \cdots, B_n\right).$$

This implies inequality (3.12) hold. □

Corollary 3.1. *Let* $n \in \mathbb{N}, n \geq 1, a_k > 0\,(k = 1, 2, \cdots, n)$*,* $B_n = \min_{1\leq k\leq n}\{ka_k\}$*, then*

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)a_k-\sum_{k=1}^{n}\left(\prod_{j=1}^{k}a_j\right)^{1/k} \geq B_n\left(\frac{4}{5}e-1\right). \tag{3.13}$$

Proof. Let $T(i) = e\sum_{k=1}^{i}\left(1-\frac{2}{3k+7}\right)\frac{1}{k}-\sum_{k=1}^{i}\frac{1}{(k!)^{1/k}}$, $i = 1, 2, \cdots, n$. Inequality (3.4)implies $\{T(i)\}_{i=1}^{n}$ is a strictly increasing sequence. According to inequality (3.11), we have

$$e\sum_{k=1}^{n}\left(1-\frac{2}{3k+7}\right)a_k-\sum_{k=1}^{n}\left(\prod_{j=1}^{k}a_j\right)^{1/k} \geq B_nT(n) \geq B_nT(1) = B_n\left(\frac{4}{5}e-1\right).$$

□

Let $n \to +\infty$, we know the following Corollary 3.2 is true.

Corollary 3.2. *If* $a_n \geq 0\,(n \in \mathbb{N}, n \geq 1)$ *with* $0 < \sum_{n=1}^{\infty} a_n < \infty$*, then*

$$\sum_{n=1}^{\infty}\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq e\sum_{n=1}^{\infty}\left(1-\frac{2}{3n+7}\right)a_n. \tag{3.14}$$

Remark 3.1. *a lot of Application of Theorem 1.1 will appear in other papers.*

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