ON SCHUR-GEOMETRICAL CONVEXITY OF FOUR-PARAMETER FAMILY OF MEANS

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Abstract. We prove that the four-parameter family of means

$$R(u,v;r,s;x,y)=\left[\frac{E(r,s;x^u,y^u}{E(r,s;x^v,y^v)}\right]^{1/(u-v)}$$

is Schur-geometrically convex in x,y if $(u+v)(r+s)\geq 0$ and Schur-geometrically concave otherwise.

In [1] the authors investigate Schur geometrical convexity of extended mean values (called also Stolarsky means, as they were introduced by Kenneth B. Stolarsky in [3])

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{r}{s}\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)} & \text{if} \quad sr(s-r)(x-y)\neq 0,\\ \left(\dfrac{1}{r}\dfrac{y^r-x^r}{\log y-\log x}\right)^{1/r} & \text{if} \quad r(x-y)\neq 0,\ s=0,\\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r-x^r)} & \text{if} \quad r=s,\ r(x-y)\neq 0,\\ \sqrt{xy} & \text{if} \quad r=s=0,\\ x & \text{if} \quad x=y\end{cases}$$

Their main result is that $E(r,s;x,y)$ is Schur-geometrically concave in variables x,y if $r+s\geq 0$ and Schur-geometrically concave otherwise. In [2] incompete analysis og Schur-geometricalconvexty of Gini means defined by

$$G(r,s;x,y)=\begin{cases}\left[\frac{x^r+y^r}{x^s+y^s}\right]^{1/(r-s)} & r\neq s\\ \exp\left(\frac{x^r\log x+y^r\log y}{x^r+y^r}\right) & r=s\end{cases}$$

is given: $G(r,s;x,y)$ is Schur-geometrically convex in x,y if $r,s\geq 0$.
Both Stolarsky and Gini means (or, more generally S-means) are members of the four-parameter family of means introduced in [4], see also [5]

$$R(u,v;r,s;x,y)=\begin{cases}\left[\frac{E(r,s;x^u,y^u}{E(r,s;x^v,y^v)}\right]^{1/(u-v)} & u\neq v\\ \exp(\frac{d}{du}\log E(r,s;x^u,y^u)) & u=v.\end{cases}$$

Letting $(u,v)=(1,0)$ we obtain Stolarsky means while $(u,v)=(2,1)$ and $(u,v)=(3/2,1/2)$ give Stolarsky and Heronian means respectively.
Let us recall some definitions: we say that $\mathbf{x}=(x_1,x_2)$ is majorized by $\mathbf{y}=(y_1,y_2)$ (and write $\mathbf{x}\prec\mathbf{y}$) if $\max(x_1,x_2)\leq\max(y_1,y_2)$ and $x_1+x_2=y_1+y_2$. For positive x_i we denote

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$\log \mathbf{x} = (\log x_1, \log x_2)$. A real function of two variables f is said to be Schur-geometrically convex if $\log \mathbf{x} \prec \log \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$.
The lemma below provides an useful characterization of Schur-convex functions:

Lemma 1.1. *Let $I \subset \mathbf{R}_+$ be an interval (possibly unbounded). A function $f : I \times I \to \mathbf{R}$ is Schur-geometrically convex if and only if for every positive a the function $f_a(x) = f(x, a/x)$ is decreasing for $x < \sqrt{a}$ (assuming that $(x, a/x) \in I \times I$).*

Proof. Suppose f is Schur-geometrically convex. For $t < s < \sqrt{a} \quad \log(s, a/s) \prec \log(t, a/t)$, hence $f(s, a/s) \leq f(t, a/s)$.
Conversely, assume $\log \mathbf{x} \prec \log \mathbf{y}$ and set $s = \min(\log x_1, \log x_2), t = \min(\log y_1, \log y_2), a = x_1 x_2$. Then $t \leq s \leq \sqrt{a}$ and monotonicity of f_a implies $f(t, a/t) \geq f(s, a/s)$. Now symmetry of f implies $f(\mathbf{y}) \geq f(\mathbf{x})$. □

R means are homogeneous of order 1 in x, y so one can easily see that $R(u, v; r, s; x, y)$ is Schur-geometrically convex (Schur-geometrically concave) in x, y if and only if $R(u, x; r, s; x, 1/x)$ decreases (increases) for $0 \leq x \leq 1$.

Theorem 1.2. *The R-means are Schur-geometrically convex if $(u + v)(r + s) \geq 0$ and Schur-geometrically concave otherwise.*

To prove it we shall need the following

Lemma 1.3. *For $t, A, B > 0$ let*

$$h(t, A, B) = At \coth At - Bt \coth Bt.$$

If $s \neq t$ and $A \neq b$, then

$$\operatorname{sgn}(h(t, A, B) - h(s, A, B)) = \operatorname{sgn}(t - s)(A - B).$$

Proof. The function $k(x) = x \coth x$ is even, so $k'(0) = 0$ and $k''(x) = \frac{2\cosh x}{\sinh^3 x}(x - \tanh x) \geq 0 \coth$. This implies that k is increasing for positive x, which is equivalent to $(A - B)h(t, A, B) > 0$.
Convexity of k means that its divided difference $\frac{h(t,A,B)}{t(A-B)}$ increases in t. The inequality

$$0 \leq \left(\frac{h(t, A, B)}{t(A - B)} \right)' = \frac{t(A - B)h'(t, A, B) - (A - B)h(t, A, B)}{t^2(A - B)^2}$$

implies $(A - B)h'(t, A, B) > 0$ for every t, so h' and $A - B$ are of the same sign. The Mean Value Theorem givex now

$$\operatorname{sgn}(h(t, A, B) - h(s, A, B)) = \operatorname{sgn}(t - s)h'(\xi, A, B) = \operatorname{sgn}(t - s)(A - B)$$

which completes the proof. □

Proof of Theorem (1.2). By Lemma (1.1) we have to show that $R(u, v; r, s; x, 1/x)$ decreases for $0 < x < 1$ if $(u + v)(r + s) \geq 0$ and increases otherwise, or, equivalently that $T(t) = \log R(u, v; r, s; e^t, e^{-t})$ increases for $t > 0$ (decreases respectively). We have

$$\begin{aligned} T(t) &= \frac{\log|\sinh urt| - \log|\sinh ust| - \log|\sinh vrt| + \log|\sinh vst|}{(u - v)(r - s)} \\ &= \frac{\log\sinh|ur|t - \log\sinh|us|t - \log\sinh|vr|t + \log\sinh|vs|t}{(u - v)(r - s)} \end{aligned}$$

and

$$T'(t) = \frac{|ur|t\coth|ur|t - |us|t\coth|us|t - |vr|t\coth|vr|t + |vs|t\coth|vs|t}{t(u-v)(r-s)}$$
$$= \frac{h(|u|,|r|t,|s|t) - h(|v|,|r|t,|s|t)}{t(u-v)(r-s)}.$$

Applying Lemma (1.3) we obtain

$$\operatorname{sgn} T'(t) = \operatorname{sgn}\frac{|u|-|v|}{u-v}\operatorname{sgn}\frac{|r|-|s|}{r-s} = \operatorname{sgn}(u+v)(r+s),$$

because $\frac{|u|-|v|}{u-v} = \frac{u^2-v^2}{(u-v)(|u|+|v|)} = \frac{u+v}{|u|+|v|}$. □

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