A GENERALIZATION OF BERGSTROM AND RADON′S INEQUALITIES IN PSEUDO-HILBERT SPACES

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Abstract. In this note we had presented two generalizations for Bergstrom and Radon′s inequalities for seminorms in pseudo-Hilbert spaces and in normed spaces. Some applications are also given.

1. Introduction

First we need to recall, see [3, 5], that a locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:

Z is complete;

there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);

there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$;

the topology of Z is compatible with the order (that is, there exists a basis of convex solid neighbourhoods of the origin);

and any monotonously decreasing sequence in Z_+ is convergent.

We shall say that a set $C \in Z$ is called *solid* if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.

As an easy example we shall consider, $Z = C$, a C^*–algebra with topology and natural involution.

Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes Z–space if it satisfies the following properties:

$\mathcal{H}$ is endowed with a Z–valued *inner product* (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h, k) \to [h, k] \in Z$ having the properties: $[h, h] \geq 0$; $[h, h] = 0$ implies $h = 0$; $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$; $[\lambda h, k] = \lambda[h, k]$; $[h, k]^* = [k, h]$;
for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h, h] \in Z$ is continuous. Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes Z*–space*.

Now, considering $Z = C$ as above, Z with $[z_1, z_2] = z_2^* z_1$ is a Loynes-Z space.

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An important result which can be used below is given in the next statement, and was proved in [5].

Let $\mathcal{H}$ and $\mathcal{K}$ be two Loynes Z-spaces. We recall that in [3, 4, 5] an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called gramian bounded, if there exists a constant $\mu > 0$ such that in the sense of order of Z holds

$$[Th, Th]_{\mathcal{K}} \leq \mu [h, h]_{\mathcal{H}}, \quad h \in \mathcal{H}. \tag{1.1}$$

We denote the class of such operators by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and $\mathcal{B}^*(\mathcal{H}, \mathcal{K}) = \mathcal{B}(\mathcal{H}, \mathcal{K}) \cap \mathcal{L}^*(\mathcal{H}, \mathcal{K})$.
We also denote the introduced norm by

$$\|T\| = \inf \left\{ \sqrt{\mu},\ \mu > 0 \text{ and satisfies (1.1)} \right\}. \tag{1.2}$$

It is known that the space $\mathcal{B}^*(\mathcal{H}, \mathcal{K})$ is a Banach space, and its involution $\mathcal{B}^*(\mathcal{H}, \mathcal{K})$ in $\mathcal{B}^*(\mathcal{K}, \mathcal{H})$ satisfies

$$\|T^*T\| = \|T\|^2, \quad T \in \mathcal{B}^*(\mathcal{H}, \mathcal{K}).$$

In particular $\mathcal{B}^*(\mathcal{H})$ is a C^*–algebra.

The following two results were presented in [3].

Lemma 1. *If p is a continuous and monotonous seminorm on Z, then $q_p(h) = (p([h,h]))^{1/2}$ is a continuous seminorm on $\mathcal{H}$.*

Proposition 1. *If $\mathcal{H}$ is a pre-Loynes Z-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of Z, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathcal{P}} = \{q_p \mid p \in \mathcal{P}\}$.*

We suppose that $mq_{p_2}(x) \leq q_{p_1}(x) \leq Mq_{p_2}(x)$, $(\forall)x \in \mathcal{H}$, with p_1, p_2 continuous and increasing seminorms on Z and M finite, $M \geq m > 0$. Because p_2 is increasing, we have

$$\frac{M^2}{m^2}\{2 + \frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\} \geq q_{p_2}^2(\frac{x}{q_{p_2}(x)} + \frac{y}{q_{p_2}(y)}) \geq 2\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)}$$

Thus, for example,

$$\frac{p_2([x,y]+[y,x])}{q_{p_2}(x)q_{p_2}(y)} \leq \frac{M^2}{m^2}\{1 + \frac{1}{2}\frac{p_1([x,y]+[y,x])}{q_{p_1}(x)q_{p_1}(y)}\}.$$

Let Z be an admissible space in the Loynes sense and $\mathcal{H}$ is a pre-Loynes Z-space. Using the Radon′s inequality we can state:

Remark 1. *If $h_k \in \mathcal{H},\ a_k > 0$ with $q_p(h_k) > 0,\ r > 0,\ k \in \{1, 2, ..., n\}$ then we shall have*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} q_p(h_k))^{r+1}}{(\sum_{k=1}^{n} a_k)^r}$$

When we take $r = 1$ a variant of Bergstrom′s inequality with seminorm is obtained.

2. The main results

Theorem 1. *For* $a_k,\ h_k \in \mathcal{H}$ *with* $q_p(h_k) > 0,\ \ r \geq 1,\ k \in \{1, 2, ..., n\},\ n \geq 2,\ n \in N$ *the following inequality takes place:*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} q_p(h_k))^{r+1}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n} \left(\frac{q_p(h_i)^{r+1}}{a_i^r} + \frac{q_p(h_j)^{r+1}}{a_j^r} - \frac{q_p(h_i + h_j)^{r+1}}{(a_i + a_j)^r}\right).$$

Proof. We shall consider the sequence,

$$d_n = \frac{q_p(h_1)^{r+1}}{a_1^r} + ... + \frac{q_p(h_n)^{r+1}}{a_n^r} - \frac{q_p(h_1 + ... + h_n)^{r+1}}{(a_1 + ... + a_n)^r}$$

and we shall prove that $(d_n)_n$ is an increasing monotonous sequence.

This fact it is indeed true if we consider

$$d_{n+1} - d_n = \frac{q_p(h_{n+1})^{r+1}}{a_{n+1}^r} - \frac{q_p(h_1 + ... + h_n + h_{n+1})^{r+1}}{(a_1 + ... + a_n + a_{n+1})^r} + \frac{q_p(h_1 + ... + h_n)^{r+1}}{(a_1 + ... + a_n)^r}$$

because q_p is seminorm which implies,

$$q_p(h_1 + ... + h_{n+1}) \leq q_p(h_1 + ... + h_n) + q_p(h_{n+1})$$

and also

$$\frac{(q_p(h_1 + ... + h_n + h_{n+1}))^{r+1}}{((a_1 + ... + a_n) + a_{n+1})^r} \leq \frac{(q_p(h_1 + ... + h_n) + q_p(h_{n+1}))^{r+1}}{((a_1 + ... + a_n) + a_{n+1})^r} \leq$$

$$\leq \frac{q_p(h_1 + ... + h_n)^{r+1}}{(a_1 + ... + a_n)^r} + \frac{q_p(h_{n+1})^{r+1}}{a_{n+1}^r}.$$

We used before the Radon′s inequality applied for $n = 2$, see [6],

$$\frac{\alpha^{r+1}}{a^r} + \frac{\beta^{r+1}}{b^r} \geq \frac{(\alpha + \beta)^{r+1}}{(a + b)^r}, \tag{1.3}$$

and we took $\alpha = q_p(h_1 + ... + h_n)$, $\beta = q_p(h_{n+1})$, $a = a_1 + ... + a_n$ and $b = a_{n+1}$.

Another proof for inequality (1.3), can be found in [1].

The sequence $(d_n)_n$ being increasing, we obtain that,

$$d_n \geq d_{n-1} \geq ... \geq d_2 \geq d_1 = 0$$

and that also means that

$$d_n \geq d_2 = \frac{q_p(h_1)^{r+1}}{a_1^r} + \frac{q_p(h_2)^{r+1}}{a_2^r} - \frac{q_p(h_1 + h_2)^{r+1}}{(a_1 + a_2)^r},\ (\forall) n \in N,\ n \geq 2.$$

The symmetry of d_n relatively to the variables a_i and h_j, $i, j \in \{1, 2, ..., n\}$ allows us to notice that

$$d_n \geq \frac{q_p(h_i)^{r+1}}{a_i^r} + \frac{q_p(h_j)^{r+1}}{a_j^r} - \frac{q_p(h_i + h_j)^{r+1}}{(a_i + a_j)^r},\ (\forall) n \in N,\ n \geq 2,\ i, j \in \{1, 2, ..., n\}.$$

∎

For $r = 1$ is obtained below a refinement of Bergstrom′s inequality.

Corollary 1. *For $a_k > 0,\ h_k \in \mathcal{H},\ k \in \{1, 2, ..., n\},\ n \geq 2,\ n \in N$ we shall obtain the following inequality:*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^2}{a_k} \geq \frac{(\sum_{k=1}^{n} q_p(h_k))^2}{(\sum_{k=1}^{n} a_k)} + \max_{1\leq i<j\leq n} \frac{(a_i + a_j)(a_j q_p^2(h_i) + a_i q_p^2(h_j)) - a_i a_j q_p^2(h_i + h_j)}{a_i a_j (a_i + a_j)}.$$

Theorem 2. *For $a_k > 0,\ x_k \in X,\ r \geq 0,\ k \in \{1, 2, ..., n\},\ n \geq 2,\ n \in N$ and every arbitrary seminorm $p,\ p : X \longrightarrow R_+$ we have:*

$$\sum_{k=1}^{n} \frac{p(x_k)^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} p(x_k))^{r+1}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n} \left(\frac{p(x_i)^{r+1}}{a_i^r} + \frac{p(x_j)^{r+1}}{a_j^r} - \frac{p(x_i + x_j)^{r+1}}{(a_i + a_j)^r}\right).$$

Corollary 2. *In fact with the above conditions, the Corollary 1 remains true for every seminorm p from a family of seminorms which defines the topology of the linear space considered instead of q_p.*

Theorem 3. *If we consider a normed space $\mathcal{H}$, $x_k \in \mathcal{H}$, $k \in \{1, 2, ..., n\}$ and with the above conditions of Theorem 1, then we have the following inequality:*

$$\sum_{k=1}^{n} \frac{||x_k||^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} ||x_k||)^{r+1}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n} \left(\frac{||x_i||^{r+1}}{a_i^r} + \frac{||x_j||^{r+1}}{a_j^r} - \frac{||x_i + x_j||^{r+1}}{(a_i + a_j)^r}\right)$$

Proof. It will be as the proof of Theorem 1, we shall only take

$$d_n = \frac{||x_1||^{r+1}}{a_1^r} + ... + \frac{||x_n||^{r+1}}{a_n^r} - \frac{||x_1 + ... + x_n||^{r+1}}{(a_1 + ... + a_n)^r}$$

and $\alpha = ||x_1 + ... + x_n||$, $\beta = ||x_{n+1}||$, $a = a_1 + ... + a_n$ and $b = a_{n+1}$ in relation (1.3).

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Remark 2. *We can consider instead of seminorm p, a norm $||.||$, in a normed space $\mathcal{H}$, $x_i \in \mathcal{H}$ and then under conditions of the above corollary we shall have,*

$$\sum_{k=1}^{n} \frac{||x_k||^2}{a_k} \geq \frac{(\sum_{k=1}^{n} ||x_k||)^2}{(\sum_{k=1}^{n} a_k)} + \max_{1\leq i<j\leq n} \frac{(a_i + a_j)(a_j ||x_i||^2 + a_i ||x_j||^2) - a_i a_j ||x_i + x_j||^2}{a_i a_j (a_i + a_j)}.$$

In what follows we shall present a generalizations of the Remark 1 concerning the Radon′s inequality for seminorms q_p.

Remark 3. *If $h_k \in \mathcal{H},\ a_k > 0,\ r > 0, s \geq 1\ k \in \{1, 2, ..., n\}$ then the following inequality takes place*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+s}}{a_k^r} \geq \frac{1}{n^{s-1}} \frac{(\sum_{k=1}^{n} q_p(h_k))^{r+s}}{(\sum_{k=1}^{n} a_k)^r}.$$

Now we shall be able to give a generalization of Theorem 1, Radon′s inequality for seminorms q_p.

Theorem 4. *For $a_k,\ h_k \in \mathcal{H}$ with $q_p(h_k) > 0,\ r \geq 0,\ s \geq 1,\ k \in \{1,2,...,n\},\ n \geq 2,\ n \in N$ the following inequality takes place:*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+s}}{a_k^r} \geq \frac{1}{n^{s-1}} \frac{(\sum_{k=1}^{n} q_p(h_k))^{r+s}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r} + \frac{q_p(h_j)^{r+s}}{a_j^r} - \frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right). \tag{2.1}$$

Proof. We shall write

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+s}}{a_k^r} = \sum_{k=1}^{n} \frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r},$$

and then applying the inequality from Theorem 1, we shall obtain,

$$\sum_{k=1}^{n} \frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{(q_p(h_i)^{\frac{r+s}{r+1}})^{r+1}}{a_i^r} + \frac{(q_p(h_j)^{\frac{r+s}{r+1}})^{r+1}}{a_j^r} - \frac{(q_p(h_i+h_j)^{\frac{r+s}{r+1}})^{r+1}}{(a_i+a_j)^r}\right).$$

This inequality becomes,

$$\sum_{k=1}^{n} \frac{(q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{a_k^r} \geq \frac{(\sum_{k=1}^{n} q_p(h_k)^{\frac{r+s}{r+1}})^{r+1}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r} + \frac{q_p(h_j)^{r+s}}{a_j^r} - \frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right).$$

Now, using the inequality from Remark 3, we shall obtain

$$\sum_{k=1}^{n} \frac{q_p(h_k)^{r+s}}{a_k^r} \geq \frac{\frac{1}{n^{s-1}}(\sum_{k=1}^{n} q_p(h_k))^{r+s}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{q_p(h_i)^{r+s}}{a_i^r} + \frac{q_p(h_j)^{r+s}}{a_j^r} - \frac{q_p(h_i+h_j)^{r+s}}{(a_i+a_j)^r}\right).$$

∎

Remark 4. *(1) In fact with the above conditions, the above theorem remains true for every seminorm p from a family of seminorms which defines the topology of the linear space considered instead of q_p:*

$$\sum_{k=1}^{n} \frac{p(x_k)^{r+s}}{a_k^r} \geq \frac{1}{n^{s-1}} \frac{(\sum_{k=1}^{n} p(x_k))^{r+s}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{p(x_i)^{r+s}}{a_i^r} + \frac{p(x_j)^{r+s}}{a_j^r} - \frac{p(x_i+x_j)^{r+s}}{(a_i+a_j)^r}\right),\ (\forall)\ x_k \in X,\ \text{with } p(x_k) > 0. \tag{2.2}$$

(2) Moreover, in every normed space X, we have under above conditions,

$$\sum_{k=1}^{n} \frac{||x_k||^{r+s}}{a_k^r} \geq \frac{1}{n^{s-1}} \frac{(\sum_{k=1}^{n} ||x_k||)^{r+s}}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n}\left(\frac{||x_i||^{r+s}}{a_i^r} + \frac{||x_j||^{r+s}}{a_j^r} - \frac{||x_i+x_j||^{r+s}}{(a_i+a_j)^r}\right),\ (\forall)\ x_k \in X. \tag{2.3}$$

(3) Finally, for $a_k,\ h_k \in \mathcal{H}$ *with* $q_p(h_k) > 0,\ r \geq 0,\ s \geq r+1,\ k \in \{1, 2, ..., n\},\ n \geq 2,\ n \in N$ *a variant of Radon′s inequality takes place:*

$$\sum_{k=1}^{n} \frac{q_p(h_k)^s}{a_k^r} \geq \frac{1}{n^{s-r-1}} \frac{(\sum_{k=1}^{n} q_p(h_k))^s}{(\sum_{k=1}^{n} a_k)^r} + \max_{1\leq i<j\leq n} \left(\frac{q_p(h_i)^s}{a_i^r} + \frac{q_p(h_j)^s}{a_j^r} - \frac{q_p(h_i+h_j)^s}{(a_i+a_j)^r} \right). \tag{2.4}$$

References

[1] Bencze M., Inequalities Connected to the Cauchy-Schwarz Inequality, *OCTOGON Mathematical Magazine*, Vol. 15, No. 1, April (2007), 58–62.

[2] Chobanyan S. A and Weron A., Banach-space-valued stationary processes and their linear prediction, Dissertations Math., 125, (1975), 1–45.

[3] Ciurdariu L., Classes of linear operators on pseudo-Hilbert spaces and applications, Part I, Monografii matematice, Tipografia Universitatii de Vest din Timişoara, 2006.

[4] Loynes R. M., Linear operators in VH-spaces, *Trans. American Math. Soc.*, 116, (1965), 167-180.

[5] Loynes R. M., On generalized positive definite functions, *Proc. London Math. Soc*, 3, (1965), 373-384.

[6] Marghidanu D., Generalizations and refinements for Bergstrom and Radon′s inequalities, *Journal of Science and Arts*, Targoviste, (2008), 57–61.

[7] Weron A. and Chobanyan S. A., Stochastic processes on pseudo-Hilbert spaces (russian), *Bull. Acad. Polon.*, Ser. Math. Astr. Phys., tom XX1, 9, (1973), 847-854.

[8] Weron A., Prediction theory in Banach spaces, *Proc. of Winter School on Probability*, Karpacz, Springer Verlag, London, 1975, 207-228.

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