

ON SCHUR-NONCONVEXITY OF GINI MEANS

ALFRED WITKOWSKI

ABSTRACT. We complete the results of Shi et al. by finding areas where Gini means are nor Schur-concave nor Schur-convex.

In [1] the authors have shown that the Gini means

$$G(r, s; x, y) = \begin{cases} \left[\frac{x^r + y^r}{x^s + y^s} \right]^{1/(r-s)} & r \neq s \\ \exp \left(\frac{x^r \log x + y^r \log y}{x^r + y^r} \right) & r = s \end{cases}$$

are Schur-convex in x, y if $r, s \geq 0$ and $r + s \geq 1$ and Schur-concave if $r + s \leq 1$ and $r \leq 0$ or $s \leq 0$. In this note we show that in all other cases Gini means are nor Schur-convex nor Schur-concave thus completing analysis of Schur-concavity.

Recall the if $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ then $\mathbf{x} \prec \mathbf{y}$ if $\max(x_i, x_2) \leq \max(y_1, y_2)$ and $x_1 + x_2 = y_1 + y_2$. A real function f is said to be Schur convex if $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$. In other words f is Schur-convex if and only if the function $f(t, a - t)$ is decreasing for $t < a/2$

Since Gini means are homogeneous of order 1 in x, y we easily see that $G(r, s; x, y)$ is Schur-convex (Schur-concave) in x, y if and only if $G(r, s; x, 1 - x)$ decreases (increases) for $0 \leq x \leq 1/2$.

Let $H(x) = G(r, s; x, 1 - x)$. An easy calculation shows that

$$(\log H(x))' \Big|_{x=1/2} = 0 \text{ and } (\log H(x))'' \Big|_{x=1/2} = r + s - 1.$$

If $r, s > 0$ and $r + s < 1$ then $\log H(x)$ increases for x close to $1/2$ and so does $H(x)$. Since $H(0) = 1 > H(1/2) = 1/2$, H cannot be monotone.

Similarly, if $s < 0$ or $r < 0$ and $r + s > 1$, then H decreases near $1/2$, $H(0) = 0 < H(1/2)$ and H cannot be monotone either.

REFERENCES

- [1] Huan-Nan Shi, Yong-Ming Jiang and Wei-Dong Jiang, *Schur-convexity and Schur-geometrically concavity of Gini means*. Comput. Math. Appl. **57** (2009), no. 2, 266–274

MIELCZARSKIEGO 4/29, 85-796 BYDGOSZCZ, POLAND
E-mail address: a4karo@gmail.com

Date: May 22, 2009.

2000 Mathematics Subject Classification. 26B25.

Key words and phrases. Gini means, Schur convexity.