ON SOME INTEGRAL INEQUALITIES RELATED TO THE CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY

N.S. BARNETT, S.S. DRAGOMIR, AND I. GOMM

ABSTRACT. Some new results that provide refinements and reverses of the Cauchy-Bunyakovsky-Schwarz (CBS) −inequality in the general setting of Measure Theory and under some boundedness conditions for the functions involved are given.

1. INTRODUCTION

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of subsets of Ω denoted by Σ and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_w^2(\Omega, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ the Hilbert space of all $\mathbb{K}$-valued functions f defined on Ω that are $2 - w-$integrable on Ω, i.e., $\int_\Omega w(x) |f(x)|^2 d\mu(x) < \infty$, where $w : \Omega \to [0, \infty)$ is a given $\mu-$measurable function on Ω.

In recent years, a number of complex-valued extensions of the celebrated results due to Polya-Szegö [6] (see also [2, p. 93]), Cassels [8], [2, p. 91] and Shisha-Mond [7] (see also [2, p. 104]) have been obtained. These provide reverses of the $CBS-$inequality. Amongst these, there is the following:

Let $\gamma, \Gamma \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ and $f, g \in L_w^2(\Omega, \mathbb{K})$ so that either

$$\int_\Omega \operatorname{Re}\left[(\Gamma g(x) - f(x))\left(\overline{f(x)} - \overline{\gamma} \cdot \overline{g(x)}\right)\right] w(x)\, d\mu(x) \geq 0 \tag{1.1}$$

or, equivalently,

$$\int_\Omega w(x) \left| f(x) - \frac{\gamma + \Gamma}{2} \right|^2 d\mu(x) \leq \frac{1}{4} |\Gamma - \gamma|^2 \int_\Omega w(x) |g(x)|^2 d\mu(x) \tag{1.2}$$

hold, then [1] (see also [5, p. 7]):

$$\begin{aligned}(0 \leq) \int_\Omega w(x) |f(x)|^2 d\mu(x) - \int_\Omega w(x) |g(x)|^2 d\mu(x) \\ - \left| \int_\Omega w(x) \overline{g(x)} d\mu(x) \right|^2 \\ \leq \frac{1}{4} |\Gamma - \gamma|^2 \left(\int_\Omega w(x) |g(x)|^2 d\mu(x) \right)^2.\end{aligned} \tag{1.3}$$

The constant $\frac{1}{4}$ is the best possible in (1.3).

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Notice that a sufficient condition for either (1.1) or (1.3) to hold is that:

$$\operatorname{Re}\left[\left(\Gamma g\left(x\right)-f\left(x\right)\right)\left(\overline{f\left(x\right)}-\overline{\gamma}\cdot\overline{g\left(x\right)}\right)\right]\geq 0 \quad \text{for } \mu-\text{a.e. } x\in\Omega. \tag{1.4}$$

With the same assumptions for γ,Γ,f,g and if $\operatorname{Re}\left(\Gamma\overline{\gamma}\right)>0$, then [4] (see also [5, p. 27]):

$$\begin{aligned}(0\leq)\int_{\Omega}w\left(x\right)\left|f\left(x\right)\right|^{2}d\mu\left(x\right)&\cdot\int_{\Omega}w\left(x\right)\left|g\left(x\right)\right|^{2}d\mu\left(x\right)\\&-\left|\int_{\Omega}w\left(x\right)\overline{g\left(x\right)}d\mu\left(x\right)\right|^{2}\\&\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}\left|\int_{\Omega}w\left(x\right)\overline{g\left(x\right)}d\mu\left(x\right)\right|^{2}.\end{aligned} \tag{1.5}$$

Here the constant $\frac{1}{4}$ is also the best possible.

Finally, if γ,Γ,f,g satisfy either (1.1) or, equivalently, (1.2) and $\Gamma\neq-\gamma$, then [3] (see also [5, p. 32]):

$$\begin{aligned}(0\leq)\left[\int_{\Omega}w\left(x\right)\left|f\left(x\right)\right|^{2}d\mu\left(x\right)\right.&\left.\cdot\int_{\Omega}w\left(x\right)\left|g\left(x\right)\right|^{2}d\mu\left(x\right)\right]^{\frac{1}{2}}\\&-\left|\int_{\Omega}w\left(x\right)\overline{g\left(x\right)}d\mu\left(x\right)\right|\\&\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\int_{\Omega}w\left(x\right)\left|g\left(x\right)\right|^{2}d\mu\left(x\right).\end{aligned} \tag{1.6}$$

The constant $\frac{1}{4}$ is again the best possible in (1.6).

We notice that, if f and g are real-valued functions and $\Gamma=M>m=\gamma>0$, then a sufficient condition for (1.1) to hold is that

$$mg\left(x\right)\leq f\left(x\right)\leq Mg\left(x\right) \quad \text{for } \mu-\text{a.e. } x\in\Omega. \tag{1.7}$$

In this case the inequalities (1.3), (1.5) and (1.6) hold with M,m instead of Γ,γ.

When μ is the discrete measure on $\Omega=\{1,\dots,n\}$, then the corresponding discrete inequalities for complex (real) numbers can be stated as well (see [5]). The details are omitted.

The main aim of this present paper is to establish other inequalities related to the $CBS-$inequality under different assumptions of boundedness for the functions involved.

2. The Results

The following result holds that provides an upper and a lower bound for the nonnegative quantity:

$$\begin{aligned}\int_{\Omega}\left|f\left(x\right)\right|^{2}w\left(x\right)d\mu\left(x\right)&\cdot\int_{\Omega}\left|g\left(x\right)\right|^{2}w\left(x\right)d\mu\left(x\right)\\&-\left[\int_{\Omega}\left|f\left(x\right)g\left(x\right)\right|w\left(x\right)d\mu\left(x\right)\right]^{2}.\end{aligned} \tag{2.1}$$

Theorem 1. *Assume that the functions $f,g:\Omega\to\mathbb{K}$, $w:\Omega\to[0,\infty)$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ are Lebesgue $\mu-$measurable on Ω and such that $\left|f\right|^{2}w$, $\left|g\right|^{2}w$, $\left|\frac{f}{g}\right|w$, $\left|\frac{g}{f}\right|w$ and w*

are also Lebesgue $\mu-$integrable on Ω. If there exist constants $0 < m < M < \infty$ such that

$$m \le |f(x)\, g(x)| \le M \quad \text{for } \mu-\text{a.e. } x \in \Omega, \tag{2.2}$$

then we have the double inequalities:

$$\begin{aligned}(0 \le) m^2 \Bigg[\int_\Omega \left|\frac{f(x)}{g(x)}\right| w(x)\, d\mu(x) \int_\Omega \left|\frac{g(x)}{f(x)}\right| w(x)\, d\mu(x) \\ - \left(\int_\Omega w(x)\, d\mu(x)\right)^2 \Bigg] \\ \le \int_\Omega |f(x)|^2 w(x)\, d\mu(x) \cdot \int_\Omega |g(x)|^2 w(x)\, d\mu(x) \\ - \left[\int_\Omega |f(x)\, g(x)|\, w(x)\, d\mu(x)\right]^2 \\ \le M^2 \Bigg[\int_\Omega \left|\frac{f(x)}{g(x)}\right| w(x)\, d\mu(x) \int_\Omega \left|\frac{g(x)}{f(x)}\right| w(x)\, d\mu(x) \\ - \left(\int_\Omega w(x)\, d\mu(x)\right)^2 \Bigg].\end{aligned} \tag{2.3}$$

Proof. We use the following elementary identity:

$$\frac{u^2+v^2}{2} - uv = \frac{1}{2} uv \left(\sqrt{\frac{u}{v}} - \sqrt{\frac{v}{u}}\right)^2$$

that holds for any $u, v > 0$, to write:

$$\begin{aligned}\frac{|f(x)|^2 |g(y)|^2 + |f(y)|^2 |g(x)|^2}{2} - |f(x)\, g(x)|\, |f(y)\, g(y)| \\ = \frac{1}{2} |f(x)\, g(x)|\, |f(y)\, g(y)| \left(\sqrt{\left|\frac{f(x)}{g(x)}\right| \cdot \left|\frac{g(y)}{f(y)}\right|} - \sqrt{\left|\frac{f(y)}{g(y)}\right| \cdot \left|\frac{g(x)}{f(x)}\right|}\right)^2,\end{aligned} \tag{2.4}$$

for $\mu-$a.e. $x, y \in \Omega$.

Since

$$\begin{aligned}\left(\sqrt{\left|\frac{f(x)}{g(x)}\right| \cdot \left|\frac{g(y)}{f(y)}\right|} - \sqrt{\left|\frac{f(y)}{g(y)}\right| \cdot \left|\frac{g(x)}{f(x)}\right|}\right)^2 \\ = \left|\frac{f(x)}{g(x)}\right| \cdot \left|\frac{g(y)}{f(y)}\right| + \left|\frac{f(y)}{g(y)}\right| \cdot \left|\frac{g(x)}{f(x)}\right| - 2\end{aligned}$$

and $0 < m \le |f(x)\, g(x)| \le M < \infty$ for $\mu-$a.e. $x \in \Omega$, then we get, from (2.4), the double inequality:

$$\begin{aligned}\frac{1}{2} m^2 \left(\left|\frac{f(x)}{g(x)}\right| \cdot \left|\frac{g(y)}{f(y)}\right| + \left|\frac{f(y)}{g(y)}\right| \cdot \left|\frac{g(x)}{f(x)}\right| - 2\right) \\ \le \frac{|f(x)|^2 |g(y)|^2 + |f(y)|^2 |g(x)|^2}{2} - |f(x)\, g(x)| \cdot |f(y)\, g(y)| \\ \le \frac{1}{2} M^2 \left(\left|\frac{f(x)}{g(x)}\right| \cdot \left|\frac{g(y)}{f(y)}\right| + \left|\frac{f(y)}{g(y)}\right| \cdot \left|\frac{g(x)}{f(x)}\right| - 2\right)\end{aligned} \tag{2.5}$$

for $\mu-$a.e. $x,y\in\Omega$.

Now, if we multiply (2.5) by $w(x)w(y)\geq 0$ and integrate over x and y, we deduce:

$$\begin{aligned}
&\frac{1}{2}m^2\left(\int_\Omega\left|\frac{f(x)}{g(x)}\right|w(x)\,d\mu(x)\cdot\int_\Omega\left|\frac{g(y)}{f(y)}\right|w(y)\,d\mu(y)\right.\\
&\qquad+\int_\Omega\left|\frac{f(y)}{g(y)}\right|w(y)\,d\mu(y)\cdot\int_\Omega\left|\frac{g(x)}{f(x)}\right|w(x)\,d\mu(x)\\
&\qquad\left.-2\int_\Omega w(x)\,d\mu(x)\int_\Omega w(y)\,d\mu(y)\right)\\
&\leq\frac{1}{2}\left[\int_\Omega|f(x)|^2w(x)\,d\mu(x)\cdot\int_\Omega|g(y)|^2w(y)\,d\mu(y)\right.\\
&\qquad\left.+\int_\Omega|f(y)|^2w(y)\,d\mu(y)\cdot\int_\Omega|g(x)|^2w(x)\,d\mu(x)\right]\\
&\qquad-\int_\Omega|f(x)g(x)|\,w(x)\,d\mu(x)\int_\Omega|f(y)g(y)|\,w(y)\,d\mu(y)\\
&\leq\frac{1}{2}M^2\left(\int_\Omega\left|\frac{f(x)}{g(x)}\right|w(x)\,d\mu(x)\cdot\int_\Omega\left|\frac{g(y)}{f(y)}\right|w(y)\,d\mu(y)\right.\\
&\qquad+\int_\Omega\left|\frac{f(y)}{g(y)}\right|w(x)\,d\mu(x)\cdot\int_\Omega\left|\frac{g(x)}{f(x)}\right|w(x)\,d\mu(x)\\
&\qquad\left.-2\int_\Omega w(x)\,d\mu(x)\int_\Omega w(y)\,d\mu(y)\right),
\end{aligned}$$

which is clearly equivalent to (2.3). ∎

In order to obtain perhaps a more convenient upper bound for the Cauchy-Bunyakovsky-Schwarz difference (2.1), we use the following Cassels' type result (see for instance [5, p. 27]):

If w is as above and $k,l:\Omega\to\mathbb{K}$ are Lebesgue $\mu-$measurable, $w|h|^2$, $w|l|^2$ are Lebesgue $\mu-$integrable on Ω and

$$0<\varphi\leq\left|\frac{h(x)}{l(x)}\right|\leq\phi<\infty\quad\text{for }\mu-\text{a.e. }x\in\Omega,\tag{2.6}$$

where φ,ϕ are constants, then (see also (1.5)):

$$\begin{aligned}
0&<\int_\Omega w(x)|h(x)|^2\,d\mu(x)\int_\Omega w(x)|l(x)|^2\,d\mu(x)\\
&\qquad-\left(\int_\Omega w(x)|h(x)l(x)|\,d\mu(x)\right)^2\\
&\leq\frac{1}{4}\cdot\frac{(\varphi-\phi)^2}{\phi\varphi}\left(\int_\Omega w(x)|h(x)l(x)|\,d\mu(x)\right)^2.
\end{aligned}\tag{2.7}$$

The following corollary holds.

Corollary 1. *Assume that f,g,w are as in Theorem 1 and that there exists a constant $M>0$ such that $|f(x)g(x)|\leq M$ for $\mu-$a.e. $x\in\Omega$. If, in addition, there exists constants $0<n<N<\infty$ such that*

$$n\leq\left|\frac{f(x)}{g(x)}\right|\leq N\quad\textit{for }\mu-a.e.\ x\in\Omega,\tag{2.8}$$

then

$$\int_{\Omega}|f(x)|^{2}w(x)\,d\mu(x)\cdot\int_{\Omega}|g(x)|^{2}w(x)\,d\mu(x) - \left[\int_{\Omega}|f(x)g(x)|\,w(x)\,d\mu(x)\right]^{2} \le \frac{1}{4}M^{2}\cdot\frac{(N-n)^{2}}{nN}\cdot\int_{\Omega}w(x)\,d\mu(x). \tag{2.9}$$

Proof. We apply the inequality (2.7) for $h=\sqrt{\left|\frac{f}{g}\right|}$, $l=\sqrt{\left|\frac{g}{f}\right|}$. Due to assumption (2.8), we have, $\sqrt{n}\le h(x)\le\sqrt{N}$ and $\frac{1}{\sqrt{N}}\le l(x)\le\frac{1}{\sqrt{n}}$ for $\mu-$a.e. $x\in\Omega$, which shows that $\varphi=n\le\frac{h(x)}{l(x)}\le N=\phi$ for $\mu-$a.e. $x\in\Omega$. We then have $\frac{(\varphi-\phi)^{2}}{\phi\varphi}=\frac{(N-n)^{2}}{nN}$ and, by (2.7),

$$\int_{\Omega}w(x)\left|\frac{f(x)}{g(x)}\right|d\mu(x)\cdot\int_{\Omega}w(x)\left|\frac{g(x)}{f(x)}\right|d\mu(x)-\left(\int_{\Omega}w(x)\,d\mu(x)\right)^{2} \le \frac{1}{4}\frac{(N-n)^{2}}{nN}\cdot\int_{\Omega}w(x)\,d\mu(x),$$

which, together with the second inequality in (2.3), provides the desired result (2.9). ∎

Now, on utilising the inequality (1.3), we can also state the following inequality for two functions h and l satisfying (2.6),

$$(0\le)\int_{\Omega}w(x)|h(x)|^{2}d\mu(x)\int_{\Omega}w(x)|l(x)|^{2}d\mu(x) - \left(\int_{\Omega}w(x)|h(x)l(x)|\,d\mu(x)\right)^{2} \le \frac{1}{4}(\varphi-\phi)^{2}\int_{\Omega}w(x)|l(x)|^{2}d\mu(x). \tag{2.10}$$

Corollary 2. *With the assumptions of Corollary 1, we have:*

$$\int_{\Omega}|f(x)|^{2}w(x)\,d\mu(x)\cdot\int_{\Omega}|g(x)|^{2}w(x)\,d\mu(x)-\left[\int_{\Omega}|f(x)g(x)|\,d\mu(x)\right]^{2} \le \frac{1}{4}M^{2}(N-n)^{2}\cdot\int_{\Omega}w(x)\left|\frac{g(x)}{f(x)}\right|^{2}d\mu(x). \tag{2.11}$$

3. Applications for $n-$tuples of Complex Numbers

On choosing the discrete measure in Theorem 1, we can state the following result for complex numbers.

Proposition 1. *Let* $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{C}^n$ *and* $p=(p_1,\dots,p_n)$ *be a probability sequence, i.e.,* $p_i\ge 0$, $i\in\{1,\dots,n\}$ *and* $\sum_{i=1}^{n}p_i=1$. *If there exist constants* $0<m<M<\infty$ *such that*

$$m\le|x_iy_i|\le M,\qquad i\in\{1,\dots,n\}, \tag{3.1}$$

then:

$$m^2\left[\sum_{i=1}^{n}p_i\left|\frac{x_i}{y_i}\right|\sum_{i=1}^{n}p_i\left|\frac{y_i}{x_i}\right|-1\right] \le \sum_{i=1}^{n}p_i|x_i|^2\sum_{i=1}^{n}p_i|y_i|^2-\left(\sum_{i=1}^{n}p_i|x_iy_i|\right)^2 \le M^2\left[\sum_{i=1}^{n}p_i\left|\frac{x_i}{y_i}\right|\sum_{i=1}^{n}p_i\left|\frac{y_i}{x_i}\right|-1\right]. \tag{3.2}$$

In addition, if the second inequality in (3.1) is satisfied and if there exist constants $0<d<D<\infty$ *such that*

$$d\le\left|\frac{x_i}{y_i}\right|\le D \qquad \text{for } \; s\in\{1,\dots,n\}, \tag{3.3}$$

then:

$$\sum_{i=1}^{n}p_i|x_i|^2\sum_{i=1}^{n}p_i|y_i|^2-\left(\sum_{i=1}^{n}p_i|x_iy_i|\right)^2\le\frac{1}{4}M^2\cdot\frac{(D-d)^2}{dD}, \tag{3.4}$$

and

$$\sum_{i=1}^{n}p_i|x_i|^2\sum_{i=1}^{n}p_i|y_i|^2-\left(\sum_{i=1}^{n}p_i|x_iy_i|\right)^2\le\frac{1}{4}M^2(D-d)^2\cdot\sum_{i=1}^{n}p_i\left|\frac{y_i}{x_i}\right|, \tag{3.5}$$

respectively.

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E-mail address: `Neil.Barnett@vu.edu.au`

E-mail address: `Sever.Dragomir@vu.edu.au`
URL: `http://www.staff.vu.edu.au/rgmia/dragomir/`

E-mail address: `Ian.Gomm@vu.edu.au`

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.