SOME PROPERTIES OF THE PSI FUNCTION

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Abstract. We obtain some complete monotonicity and strongly complete monotonicity properties of the psi function, which extends some known results due to S.-L.Qiu and M.Vuorinen.

1. Introduction

We recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $n \geq 0$. Dubourdien [4] pointed out that if a non-constant function f is completely monotonic, then strict inequality holds in (1). It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$, see [11, p. 161]. In 1974, C. H. Kimberling [5] established the following property of completely monotonic functions: If f is continuous on $[0, \infty)$ and completely monotonic on $(0, \infty)$ and satisfies $0 < f(x) \leq 1$ for all $x \geq 0$, then $\log(f)$ is super-additive on $[0, \infty)$. We recall that a function g is said to be super-additive on an interval I if

$$g(x) + g(y) \leq g(x + y) \quad \text{for all} \quad x, y \in I \quad \text{with} \quad x + y \in I.$$

A function f is said to be star-shaped on $(0, \infty)$ if

$$f(ax) \leq af(x) \tag{2}$$

is valid for all $x > 0$ and for all $a \in (0, 1)$. These functions have been investigated intensively by A. M. Bruckner and E. Ostrow [2]. It is well known that star-shaped functions are super-additive. Indeed, from (2) we obtain $f(x) \leq \frac{x}{x+y} f(x + y)$ and $f(y) \leq \frac{x}{x+y} f(x + y)$; summing leads to $f(x) + f(y) \leq f(x + y)$.

In 1989, S. Y. Trimble et al. [10] introduced an interesting subclass of the completely monotonic functions. A function g is called strongly completely monotonic on $(0, \infty)$ if

$$x \mapsto (-1)^n x^{n+1} g^{(n)}(x)$$

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is nonnegative and decreasing on $(0, \infty)$ for $n = 0, 1, 2, \ldots$. The authors showed that these functions have a close connection to star-shaped functions. Indeed, one of their results states: If g is strongly completely monotonic on $(0, \infty)$ and $g \not\equiv 0$, then $1/g$ is star-shaped.

The following proposition contains a simple characterization of strongly completely monotonic functions and its proof is obtained by a direct application of the definitions given above.

Proposition 1. *A function $f(x)$ is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic.*

In [10] the authors gave another characterization of strongly completely monotonic functions.

Proposition 2. *The function $f(x)$ is strongly completely monotonic if and only if*

$$f(x) = \int_0^\infty p(t)e^{-xt}dt,$$

where $p(t)$ is nonnegative and increasing and the integral converges for all $x > 0$.

An extension of Propositions 1 and 2 is given in [6, Theorem 1.3].

Euler's gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t \quad (x > 0)$$

is one of the most important functions in analysis and its applications. The logarithmic derivative of the gamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as psi function or digamma function. The derivatives $\psi^{(n)}$ of the psi function are called polygamma functions in the literature.

In 2004, S.-L. Qiu and M. Vuorinen [9] obtained some monotonicity and convexity properties of the gamma and psi functions, from which several asymptotically sharp inequalities follow. Applying these properties, the authors improve some well-known results for the volume Ω_n of the unit ball $B^n \subset \mathbb{R}^n$, the surface area ω_{n-1} of the unit sphere S^{n-1}, and some related constants. The following Theorem A about some properties of $\psi(x)$ was established in [9, Theorem 2.1].

Theorem A. (i) *The function $h_1(x) = \psi(x + \frac{1}{2}) - \psi(x) - \frac{1}{2x}$ is strictly decreasing and convex from $(0, \infty)$ onto $(0, \infty)$. Moreover, $h_2(x) = h_1(1/x)$ is convex on $(0, \infty)$.*

(ii) *The function $h_3(x) = xh_1(x)$ is strictly decreasing from $(0, \infty)$ onto $(0, \frac{1}{2})$.*

(iii) *The function $h_4(x) = x^2h_1(x)$ is strictly increasing from $(0, \infty)$ onto $(0, \frac{1}{8})$.*

(iv) *The function $h_5(x) = x^2[\psi(x) - \log x] + \frac{x}{2}$ is strictly decreasing and convex from $(0, \infty)$ onto $(-\frac{1}{12}, 0)$.*

(v) *The functions $\eta(x) = \frac{1}{2}\psi'(x) - \log x + \psi(x)$ and $\theta(x) = \psi(x + 1) - \log x - \frac{1}{2}\psi'(x + 1)$ are both strictly decreasing from $(0, \infty)$ onto $(0, \infty)$*

We extend the result given by Qiu and Vuorinen as follows:

Theorem 1. (i) *The function $h_1(x) = \psi(x + \frac{1}{2}) - \psi(x) - \frac{1}{2x}$ is strongly completely monotonic on $(0, \infty)$, and the function $h_3(x) = xh_1(x)$ is completely monotonic on $(0, \infty)$. Moreover, $h_2(x) = h_1(1/x)$ is convex on $(0, \infty)$.*

(ii) *The function $h_4(x) = x^2h_1(x)$ is a so-called Bernstein function on $(0, \infty)$, that is, $h_4 > 0$ and h_4' is completely monotonic on $(0, \infty)$.*

(iii) *The function* $H(x) = x^2[\psi(x) - \log x] + \frac{x}{2} + \frac{1}{12}$ *is completely monotonic on* $(0, \infty)$.

(iv) *The function* $\theta(x) = \psi(x+1) - \log x - \frac{1}{2}\psi'(x+1)$ *is completely monotonic on* $(0, \infty)$*, and* $\eta(x) = \frac{1}{2}\psi'(x) - \log x + \psi(x)$ *is strongly completely monotonic on* $(0, \infty)$.

2. Proofs of Theorems

Proof of Theorem 1. (i) The author [3, Theorem 1] proved that let $a \geq 0, b > 0$ be given real numbers with $0 < 1 - b + a < 1$, then for all real numbers α, the function $f_\alpha(x) = (x + a)^\alpha[\psi(x + b) - \psi(x + a) - \frac{b-a}{x+a}]$ is completely monotonic on $(-a, \infty)$ if and only if $\alpha \leq 1$. In particular, the function $h_3(x)$ is completely monotonic on $(0, \infty)$. By Proposition 2, the function $h_1(x)$ is strongly completely monotonic on $(0, \infty)$.

M. Merkle [8, Lemma 1] proved that a function $x \mapsto f(1/x)$ is convex on $(0, \infty)$ if and only if $x \mapsto xf(x)$ is convex on $(0, \infty)$. Hence, the convexity of $x \mapsto xh_1(x)$ implies the convexity of $x \mapsto h_1(1/x)$.

(ii) Using the representations

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt \quad \text{(see [1, p. 259])} \tag{3}$$

and

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt, \tag{4}$$

we obtain

$$h_4(x) = x^2 \int_0^\infty \delta(t) e^{-xt} dt,$$

where

$$\delta(t) = \frac{1}{2} - \frac{1}{e^{t/2} + 1}, \quad t > 0.$$

It is easy to see that the function δ is strictly increasing on $(0, \infty)$ with

$$\delta(t) > \lim_{t \to 0} \delta(t) = 0.$$

Hence, $h_4(x) > 0$ for $x > 0$.

By using the convolution theorem for Laplace transforms, we have for $n \geq 1$,

$$\begin{aligned}
\frac{(-1)^n}{x^2} h_4^{(n)}(x) &= \frac{(-1)^n}{x^2} \sum_{k=0}^{n} \binom{n}{k} (x^2)^{(n-k)} \left(\int_0^\infty \delta(t) e^{-xt} dt \right)^{(k)} \\
&= \int_0^\infty t^n \delta(t) e^{-xt} dt - \frac{2n}{x} \int_0^\infty t^{n-1} \delta(t) e^{-xt} dt \\
&\qquad + \frac{n(n-1)}{x^2} \int_0^\infty t^{n-2} \delta(t) e^{-xt} dt \\
&= \int_0^\infty t^n \delta(t) e^{-xt} dt - 2n \int_0^\infty e^{-xt} dt \int_0^\infty t^{n-1} \delta(t) e^{-xt} dt \\
&\qquad + n(n-1) \int_0^\infty t e^{-xt} dt \int_0^\infty t^{n-2} \delta(t) e^{-xt} dt \\
&= \int_0^\infty t^n \delta(t) e^{-xt} dt - 2n \int_0^\infty \left[\int_0^t u^{n-1} \delta(u) du \right] e^{-xt} dt \\
&\qquad + n(n-1) \int_0^\infty \left[\int_0^t (t-u) u^{n-2} \delta(u) du \right] e^{-xt} dt \\
&= \int_0^\infty \chi_n(t) e^{-xt} dt,
\end{aligned} \tag{5}$$

where

$$\chi_n(t) = t^n \delta(t) - 2n \int_0^t u^{n-1} \delta(u) du + n(n-1) \int_0^t (t-u) u^{n-2} \delta(u) du.$$

Differentation yields

$$\chi_n'(t) = t^n \delta'(t) - n t^{n-1} \delta(t) + n(n-1) \int_0^t u^{n-2} \delta(u) du \quad \text{and} \quad \chi_n''(t) = t^n \delta''(t).$$

Easy computation gives

$$\delta''(t) = -\frac{(e^{t/2} - 1) e^{t/2}}{4(e^{t/2} + 1)^3} < 0 \quad \text{for} \quad t > 0,$$

and therefore, we have

$$\chi_n''(t) < 0 \Longrightarrow \chi_n'(t) < \chi_n'(0) = 0 \Longrightarrow \chi_n(t) < \chi_n(0) = 0,$$

so that (5) implies that

$$(-1)^n h_4^{(n)}(x) < 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad n \geq 1.$$

(iii) Since the function $h_5(x) = x^2[\psi(x) - \log x] + \frac{x}{2}$ is strictly decreasing from $(0, \infty)$ onto $(-\frac{1}{12}, 0)$, we obtain

$$H(x) > 0 \quad \text{for} \quad x > 0. \tag{6}$$

Using the representations (3), (4) and

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \tag{7}$$

see [1, p. 230], we obtain

$$H(x) = x^2 \int_0^\infty \varphi(t) e^{-xt} dt + \frac{1}{12},$$

where

$$\varphi(t) = \frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2}, \quad t > 0.$$

For $n \geq 1$, we have

$$\begin{aligned}
\frac{(-1)^n}{x^2} H^{(n)}(x) &= \frac{(-1)^n}{x^2} \sum_{k=0}^{n} \binom{n}{k} (x^2)^{(n-k)} \left(\int_0^\infty \varphi(t) e^{-xt} dt \right)^{(k)} \\
&= \int_0^\infty t^n \varphi(t) e^{-xt} dt - \frac{2n}{x} \int_0^\infty t^{n-1} \varphi(t) e^{-xt} dt \\
&\qquad + \frac{n(n-1)}{x^2} \int_0^\infty t^{n-2} \varphi(t) e^{-xt} dt \\
&= \int_0^\infty t^n \varphi(t) e^{-xt} dt - 2n \int_0^\infty e^{-xt} dt \int_0^\infty t^{n-1} \varphi(t) e^{-xt} dt \\
&\qquad + n(n-1) \int_0^\infty t e^{-xt} dt \int_0^\infty t^{n-2} \varphi(t) e^{-xt} dt \\
&= \int_0^\infty t^n \varphi(t) e^{-xt} dt - 2n \int_0^\infty \left[\int_0^t u^{n-1} \varphi(u) du \right] e^{-xt} dt \\
&\qquad + n(n-1) \int_0^\infty \left[\int_0^t (t-u) u^{n-2} \varphi(u) du \right] e^{-xt} dt \\
&= \int_0^\infty \lambda_n(t) e^{-xt} dt,
\end{aligned} \tag{8}$$

where

$$\lambda_n(t) = t^n \varphi(t) - 2n \int_0^t u^{n-1} \varphi(u) du + n(n-1) \int_0^t (t-u) u^{n-2} \varphi(u) du.$$

Differentation yields

$$\lambda_n'(t) = t^n \varphi'(t) - n t^{n-1} \varphi(t) + n(n-1) \int_0^t u^{n-2} \varphi(u) du \quad \text{and} \quad \lambda_n''(t) = t^n \varphi''(t).$$

Easy computation gives

$$\begin{aligned}
t^3 (e^t - 1)^3 \varphi''(t) &= 2(e^t - 1)^3 - t^3 (e^{2t} + e^t) \triangleq \mu(t), \\
\mu'(t) &= 12 e^{2t} - (2t^3 + 9t^2 + 6t + 12) e^t - 3t^2 - 6t \\
&= \sum_{k=5}^{\infty} [12 \cdot 2^n - n(n+1)(2n+1) - 12] \frac{t^k}{k!} > 0, \quad t > 0,
\end{aligned}$$

and therefore, we have

$$\begin{aligned}
\mu(t) > \mu(0) = 0 \Longrightarrow \varphi''(t) > 0 \Longrightarrow \lambda_n''(t) > 0 \Longrightarrow \lambda_n'(t) > \lambda_n'(0) = 0 \\
\Longrightarrow \lambda_n(t) > \lambda_n(0) = 0.
\end{aligned}$$

Hence, we obtain by (6) and (8) that

$$(-1)^n H^{(n)}(x) > 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad n \geq 0.$$

(iv) Using the representations (3), (7) and

$$\psi'(x) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-xt} dt, \tag{9}$$

see [1, p. 260], we get

$$\theta(x) = \int_0^\infty \nu(t)e^{-xt}dt, \tag{10}$$

where

$$\nu(t) = \frac{1}{t} - \frac{\frac{1}{2}t+1}{e^t-1} = \frac{e^t-1-t-\frac{1}{2}t^2}{t(e^t-1)} > 0, \quad t > 0,$$

so that (10) implies that the function θ is completely monotonic on $(0,\infty)$.

Using the representations (3), (7) and (9), we obtain

$$\eta(x) = \int_0^\infty \phi(t)e^{-xt}dt,$$

where

$$\phi(t) = \frac{\frac{1}{2}t-1}{1-e^{-t}} + \frac{1}{t}, \quad t > 0.$$

Differentation yields

$$\begin{aligned} t^2(e^t-1)^2\phi'(t) &= \left(\frac{1}{2}t^2-1\right)e^{2t} - \left(\frac{1}{2}t^3 - \frac{1}{2}t^2 - 2\right)e^t - 1 \\ &= \sum_{n=4}^\infty \left\{2^n\left[\frac{n(n-1)}{8} - 1\right] - \frac{n(n-1)(n-3)}{2} + 2\right\}\frac{t^n}{n!} > 0 \end{aligned}$$

for $t > 0$, and therefore, the function ϕ is strictly increasing on $(0,\infty)$ with

$$\phi(t) > \lim_{t\to 0}\phi(t) = 0.$$

By Proposition 2, the function $\eta(x)$ is strongly completely monotonic on $(0,\infty)$. □

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