# SUPERADDITIVITY AND SUPERMULTIPLICITY OF TWO FUNCTIONALS ASSOCIATED WITH THE STIELTJES INTEGRAL

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ABSTRACT. The superadditivity and supermultiplicity of two functionals associated with the Stieltjes integral are established. Some applications are provided.

## 1. INTRODUCTION

In the Theory of the Stieltjes Integral for scalar functions, it is well known that if $f : [a, b] \to \mathbb{R}$ ($\mathbb{C}$) is continuous and $u : [a, b] \to \mathbb{R}$ ($\mathbb{C}$) is of bounded variation, then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and the following sharp inequality holds true

$$\left| \int_a^b f(t)\,du(t) \right| \leq \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u), \tag{1.1}$$

where $\bigvee_a^b (u)$ denotes the total variation of $u$ on $[a, b]$, we recall that

$$\bigvee_a^b (u) = \sup \left\{ \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|,\ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \right\}.$$

The inequality (1.1) plays an important role in obtaining various sharp bounds for the approximation error of the Stieltjes integral by simpler quantities such as:

$$f(x)[u(b) - u(a)], \qquad \text{(see [2], [3], [1])}$$
$$f(b)[u(b) - u(x)] + f(a)[u(x) - u(a)] \qquad \text{(see [4], [1])}$$

and

$$\frac{1}{b-a}[u(b) - u(a)] \int_a^b f(t)\,dt \qquad \text{(see [5], [6])},$$

where $x \in [a, b]$.

The main aim of this paper is to point out some superadditivity and supermultiplicity properties of two functionals that can be naturally associated with the fundamental inequality (1.1). Some applications are provided as well.

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## 2. A Superadditivity Property

For a continuous function $f : [a,b] \to \mathbb{R}$ and a function of bounded variation $u : [a,b] \to \mathbb{R}$ we define the following functional

$$\Psi(f,u;[a,b]) := \max_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b (u) - \left| \int_a^b f(t)\, du(t) \right|. \tag{2.1}$$

Due to the properties of the Stieltjes integral, the functional $\Psi$ is well defined and nonnegative.

The following properties concerning its behaviour as a functional of an interval may be stated.

**Theorem 1.** *Assume that $f : [a,b] \to \mathbb{R}$ is continuous and $u : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $c \in (a,b)$ we have*

$$(0 \le)\, \Psi(f,u;[a,c]) + \Psi(f,u;[c,b]) \le \Psi(f,u;[a,b]), \tag{2.2}$$

*i.e., $\Psi(f,u;\cdot)$ is superadditive as a functional of an interval.*

*If $[c,d] \subseteq [a,b]$, then*

$$(0 \le)\, \Psi(f,u;[c,d]) \le \Psi(f,u;[a,b]), \tag{2.3}$$

*i.e., $\Psi(f,u;\cdot)$ is monotnic nondecreasing as a functional of an interval. Moreover, we have:*

$$\sup\{\Psi(f,u;[c,d]) | [c,d] \subseteq [a,b]\} = \Psi(f,u;[a,b]). \tag{2.4}$$

*Proof.* Let $c \in (a,b)$. Then

$$\begin{aligned}
\Psi(f,u;[a,b]) &= \max\left\{ \max_{t\in[a,c]} |f(t)|, \max_{t\in[c,b]} |f(t)| \right\} \left[ \bigvee_a^c (f) + \bigvee_c^b (f) \right] \\
&\quad - \left| \int_a^c f(t)\, du(t) + \int_c^b f(t)\, du(t) \right| \\
&\ge \max_{t\in[a,c]} |f(t)| \cdot \bigvee_a^c (f) + \max_{t\in[c,b]} |f(t)| \cdot \bigvee_c^b (f) \\
&\quad - \left| \int_a^c f(t)\, du(t) \right| - \left| \int_c^b f(t)\, du(t) \right| \\
&= \Psi(f,u;[a,c]) + \Psi(f,u;[c,b])
\end{aligned}$$

and the superadditivity of $\Psi(f,u;\cdot)$ is thus proven.

Now, let $a < c < d < b$. Then by the superadditivity of $\Psi(f,u;\cdot)$ we have

$$\Psi(f,u;[a,b]) - \Psi(f,u;[c,d]) \ge \Psi(f,u;[a,c]) + \Psi(f,u;[d,b]) \ge 0,$$

which proves the monotonicity property.

The representation (2.4) follows from (2.3) and the theorem is thus proved. ∎

In [2], in order to approximate the Stieltjes integral $\int_a^b f(t)\, du(t)$ with the quantity $f(x)\,[u(b) - u(a)]$, the author has considered the *Ostrowski error functional*

$$\Theta(f,u;[a,b],x) := f(x)[u(b) - u(a)] - \int_a^b f(t)\, du(t)$$

and showed that:

$$|\Theta(f,u;[a,b],x)| \leq \max_{t\in[a,b]}|f(t)-f(x)|\bigvee_a^b(u), \tag{2.5}$$

provided $f$ is continuous on $[a,b]$ and $u$ is of bounded variation on $[a,b]$.

Now, if we define

$$\Psi_\Theta(f,u;[a,b],x) := \max_{t\in[a,b]}|f(t)-f(x)|\bigvee_a^b(u) - |\Theta(f,u;[a,b],x)|,$$

then, on utilising the monotonicity property of $\Psi(f,u;\cdot)$, we can show that the absolute value of the error $\Theta(f,u;\cdot,x)$ is closer to the theoretical bound $\max_{t\in[\cdot,\cdot]}|f(t)-f(x)|\bigvee_{\cdot}^{\cdot}(u)$ for smaller intervals than that for larger ones. More specifically, we can state the following corollary.

**Corollary 1.** *Let $f$ be a continuous function on $[a,b]$ and $u$ a function of bounded variation on $[a,b]$. If $[c,d]\subseteq[a,b]$, then*

$$\Psi_\Theta(f,u;[c,d],x) \leq \Psi_\Theta(f,u;[a,b],x) \tag{2.6}$$

*for any $x\in[c,d]$.*

In [4], in order to approximate the Stieltjes integral $\int_a^b f(t)\,du(t)$ with the generalised trapezoidal rule $[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)$, where $f$ is a function of bounded variation while $u$ is continuous on $[a,b]$, the authors considered the *Generalised Trapezoidal error functional*

$$T(f,u;[a,b],x) := [u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t)$$

and showed that

$$|T(f,u;[a,b],x)| \leq \max_{t\in[a,b]}|u(t)-u(x)|\bigvee_a^b(f). \tag{2.7}$$

Now, if we consider the functional

$$\Psi_T(f,u;[a,b],x) := \max_{t\in[a,b]}|u(t)-u(x)|\bigvee_a^b(f) - |T(f,u;[a,b],x)|,$$

then the following corollary may be stated as well:

**Corollary 2.** *Let $f$ be a function of bounded variation on $[a,b]$ and $u$ a continuous function on $[a,b]$. If $[c,d]\subseteq[a,b]$, then*

$$(0\leq)\,\Psi_T(f,u;[c,d],x) \leq \Psi_T(f,u;[a,b],x) \tag{2.8}$$

*for any $x\in[c,d]$.*

## 3. A Supermultiplicity Property

Now, consider the new functional

$$\Phi(f,u;[a,b]) := \left[\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\bigvee_a^b(u) - \left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^{(b-a)},$$

which is well defined for continuous functions $f : [a,b] \to \mathbb{R}$ and functions of bounded variation $u : [a,b] \to \mathbb{R}$. We observe that

$$\Phi(f,u;[a,b]) = \left[\frac{1}{b-a}\Psi(f,u;[a,b])\right]^{(b-a)}. \tag{3.1}$$

The following result may be stated.

**Theorem 2.** *Assume that $f : [a,b] \to \mathbb{R}$ is continuous and $u : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $c \in (a,b)$ we have*

$$\Phi(f,u;[a,b]) \geq \Phi(f,u;[a,c]) \cdot \Phi(f,u;[c,b]), \tag{3.2}$$

*i.e., $\Phi(f,u;\cdot)$ is supermultiplicative as a function of an interval.*

*Proof.* Define the new functional

$$\eta(f,u;[a,b]) := \frac{1}{b-a} \cdot \Psi(f,u;[a,b]). \tag{3.3}$$

Utilising the superadditivity property of the functional $\Psi(f,u;\cdot)$ we have for any $c \in (a,b)$ that

$$(b-a)\,\eta(f,u;[a,b]) \geq (b-c)\,\eta(f,u;[c,b]) + (c-a)\,\eta(f,u;[a,c]),$$

which gives

$$\eta(f,u;[a,b]) \geq \frac{b-c}{b-a} \cdot \eta(f,u;[c,b]) + \frac{c-a}{b-a} \cdot \eta(f,u;[a,c]). \tag{3.4}$$

Now, on making use of the *weighted arithmetic mean – geometric mean inequality*:

$$p\alpha + q\beta \geq \alpha^p \cdot \beta^q, \qquad \alpha,\beta > 0 \tag{3.5}$$

$p,q > 0$ with $p+q=1$, for the choices

$$\alpha = \eta(f,u;[c,b]), \quad \beta = \eta(f,u;[a,c]), \quad p = \frac{b-c}{b-a}, \quad q = \frac{c-a}{b-a},$$

we also get:

$$\frac{b-c}{b-a} \cdot \eta(f,u;[c,b]) + \frac{c-a}{b-a} \cdot \eta(f,u;[a,c]) \geq [\eta(f,u;[c,b])]^{\frac{b-c}{b-a}} \cdot [\eta(f,u;[a,c])]^{\frac{c-a}{b-a}}. \tag{3.6}$$

Combining the inequality (3.4) and (3.6) we can state that

$$\eta(f,u;[a,b]) \geq [\eta(f,u;[c,b])]^{\frac{b-c}{b-a}} \cdot [\eta(f,u;[a,c])]^{\frac{c-a}{b-a}}. \tag{3.7}$$

Taking the power $b-a$ in (3.7), we obtain the result (3.2). ∎

## 4. Applications

If $g : [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$ then $u(t) = \int_a^t g(s)\,ds$ is differentiable on $(a,b)$ and the functionals $\Psi$ and $\Phi$ become

$$\tilde{\Psi}(f,g;[a,b]) := \max_{t\in[a,b]} |f(t)| \cdot \int_a^b |g(t)|\,dt - \left|\int_a^b f(t)\,g(t)\,dt\right|, \tag{4.1}$$

and

$$\tilde{\Phi}(f,g;[a,b]) \tag{4.2}$$

$$:= \left[\max_{t\in[a,b]}|f(t)|\cdot\frac{1}{b-a}\int_a^b|g(t)|\,dt-\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|\right]^{(b-a)}.$$

Obviously $\tilde{\Psi}$ remains *superadditive and monotonic nondecreasing as a functional* of an interval while $\tilde{\Phi}$ inherits the supermultiplicity property of $\Phi$.

Let us recall the following means:

$$\begin{aligned}
\text{Arithmetic mean} &: \quad A(a,b)=\frac{a+b}{2},\\
\text{Geometric mean} &: \quad G(a,b)=\sqrt{ab},\\
\text{Harmonic mean} &: \quad H(a,b)=\frac{2}{\frac{1}{a}+\frac{1}{b}},\\
\text{Logarithmic mean} &: \quad L(a,b)=\frac{b-a}{\ln b-\ln a},\ b\neq a;\\
\text{Identric mean} &: \quad I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\ b\neq a;\\
\text{p-Logarithmic mean} &: \quad L_p(a,b)=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\\
p &\in \mathbb{R}\backslash\{-1,0\},\, b\neq a;
\end{aligned}$$

with $a,b>0$.

It is well known that $L_p$ is monotonic nondecreasing over $p\in\mathbb{R}$ with $L_{-1}:=L$ and $L_0:=I$. In particular, we have the following inequalities:

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b). \tag{4.3}$$

If we consider $u(t)=t^p$, $p\in\mathbb{R}\backslash\{-1,0\}$, $u:[a,b]\to\mathbb{R}$, $0<a<b$, then obviously

$$\frac{1}{b-a}\int_a^b u(t)\,dt=L_p^p(a,b).$$

If $u(t)=\frac{1}{t}$, $t\in[a,b]$, $0<a<b$, then

$$\frac{1}{b-a}\int_a^b u(t)\,dt=\frac{1}{L(a,b)},$$

while for $u(t)=\ln t$, $t\in[a,b]$, $0<a<b$, we have

$$\frac{1}{b-a}\int_a^b u(t)\,dt=\ln[I(a,b)].$$

If we choose above $g(t)=\frac{1}{t}$, $f(t)=t^p$ with $p\in\mathbb{R}\backslash\{0,1\}$ and observing that $0<a<b$, we have:

$$\max_{tse[a,b]}|f(t)|=\max\{a^p,b^p\},\qquad \int_a^b g(t)\,dt=\frac{b-a}{L(a,b)}$$

and

$$\int_a^b f(t)\,g(t)\,dt=L_{p-1}^{p-1}(a,b)(b-a),$$

then we deduce that

$$v_p([a,b]) := \left[\frac{\max\{a^p, b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)\right](b-a), \qquad p \in \mathbb{R}\setminus\{0,1\}. \tag{4.4}$$

is *superadditive and monotonic nondecreasing* as a functional on positive intervals while

$$w_p([a,b]) := \left[\frac{\max\{a^p, b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)\right]^{(b-a)}, \qquad p \in \mathbb{R}\setminus\{0,1\} \tag{4.5}$$

is *supermultiplicative* as a functional on positive intervals.

Similar results may be stated for other choices of $f$ and $g$. However, the details are omitted.

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