# SOME NEW HILBERT-PACHPATTE'S INEQUALITIES FOR A FINITE NUMBER OF NONNEGATIVE SEQUENCES

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Abstract. Some new Hilbert-Pachpatte's inequalities for a finite number of nonnegative sequences are established in this paper. Integral analogues of main results are also given.

## 1. Introduction

Let $p \geq 1$, $q \geq 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \cdots, k$ and $n = 1, 2, \cdots, r$, where $k$ and $r$ are natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \leq C(p,q,k,r)\left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^2\right\}^{\frac{1}{2}} \times \left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^2\right\}^{\frac{1}{2}}, \tag{1.1}$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p,q,k,r) = \frac{1}{2}pq\sqrt{kr}$.

An integral analogue of (1.1) is given in the following result.

Let $p \geq 1$, $q \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0,x)$, $\tau \in (0,y)$, where $x, y$ are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0,x)$, $t \in (0,y)$. Then

$$\int_0^x\int_0^y \frac{F^p(s)G^q(t)dsdt}{s+t} \leq D(p,q,x,y)\left\{\int_0^x (x-s)(F^{p-1}(s)f(s))^2ds\right\}^{\frac{1}{2}} \times \left\{\int_0^y (y-t)(G^{q-1}(t)g(t))^2dt\right\}^{\frac{1}{2}}, \tag{1.2}$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p,q,x,y) = \frac{1}{2}pq\sqrt{xy}$.

Inequalities (1.1) and (1.2) are the well known Hilbert-Pachpatte's inequalities[1], which gave new estimates on Hilbert type inequalities[2]. It is well known that Hilbert-Pachpatte's inequalities play a dominant role in analysis, so the literature on such inequalities and their applications is vast[3-9].

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Handley et al.[10] gave some Hilbert's inequalities for a finite number of nonnegative sequences as follows.

Let $\{a_{i,m_i}\}$ $(i = 1, 2, \cdots, n)$ be $n$ sequences of nonnegative sequences of real numbers defined for $m_i = 1, 2, \cdots, k_i$ with $a_{1,0} = a_{20} = \cdots = a_{n,0} = 0$, and let $p_{i,m_i}$ be $n$ sequences of positive real numbers defined for $m_i = 1, 2, \cdots, k_i$, where $k_i$ $(i = 1, 2, \cdots, n)$ are natural numbers. Set $P_{i,m_i} = \sum_{s_i=1}^{m_i} p_{i,s_i}$. Let $\phi_i$ $(i = 1, 2, \cdots, n)$ be $n$ real-valued nonnegative convex and submultiplicative functions defined on $R_+ = [0, \infty)$. Let $\alpha_i \in (0, 1)$ and $\alpha'_i = 1 - \alpha_i$ $(i = 1, 2, \cdots, n)$, $\alpha = \sum_{i=1}^n \alpha_i$ and $\alpha' = \sum_{i=1}^n \alpha'_i = n - \alpha$. Then

$$(1.3)\quad \sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \phi_i(a_{i,m_i})}{(\sum_{i=1}^n {\alpha'}_i m_i)^{\alpha'}} \le M(k_1, \cdots, k_n) \times \prod_{i=1}^n \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left[ p_{i,m_i} \phi_i \left( \frac{\nabla a_{i,m_i}}{p_{i,m_i}} \right) \right]^{\frac{1}{\alpha_i}} \right\}^{\alpha_i},$$

where $M(k_1, \cdots, k_n) = \frac{1}{\alpha'^{\alpha'}} \prod_{i=1}^n \left\{ \sum_{m_i=1}^{k_i} \left[ \left( \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right) \right]^{\frac{1}{\alpha'_i}} \right\}^{\alpha'_i}$, and $\nabla a_{i,m_i} = a_{i,m_i} - a_{i,m_i-1}$ $(i = 1, 2, \cdots, n)$.

An integral analogue of (1.3) is given in the following result.

Let $f_i \in C^1[[o, x_i], R_+]$, $i = 1, 2, \cdots, n$, with $f_i(0) = 0$ $(i = 1, 2, \cdots, n)$. Let $p_i(\sigma_i)$ be n positive functions defined for $\sigma_i \in [0, x_i]$ $(i = 1, 2, \cdots, n)$. Set $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$ for $s_i \in [0, x_i]$, where $x_i$ are positive real numbers. Let $\phi_i$, $\alpha_i$, $\alpha'_i$, $\alpha$, $\alpha'$, be as in the above result. Then

$$(1.4)\quad \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(f_i(s_i))}{(\sum_{i=1}^n {\alpha'}_i s_i)^{\alpha'}} ds_1 \cdots ds_n \le L(x_1, \cdots, x_n) \times \prod_{i=1}^n \left\{ \int_0^{x_i} (x_i - s_i) \left[ p_i(s_i) \phi_i \left( \frac{f'_i(s_i)}{p_i(s_i)} \right) \right]^{\frac{1}{\alpha_i}} ds_i \right\}^{\alpha_i},$$

where $L(x_1, \cdots, x_n) = \frac{1}{\alpha'^{\alpha'}} \prod_{i=1}^n \left\{ \int_0^{x_i} \left[ \left( \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right) \right]^{\frac{1}{\alpha'_i}} ds_i \right\}^{\alpha'_i}$.

The purpose of the present paper, motivated by Handley et al.[10], is to derive some general versions of inequalities (1.1) and (1.2) and similar to (1.3) and (1.4). We also establish the two independent variable versions of the inequalities (1.1) and (1.2) for a finite number of nonnegative sequences.

## 2. Main results

Now we give our results as follows in this paper.

**Theorem 1.** *Let* $q_i \ge 1$, $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j \ne i}^n p_j$ $(i = 1, 2, \cdots, n)$. *Let* $\{a_{i,m_i}\}$ $(i = 1, 2, \cdots, n)$ *be* $n$ *sequences of nonnegative sequences of real numbers defined for* $m_i = 1, 2, \cdots, k_i$, *where* $k_i$ $(i = 1, 2, \cdots, n)$ *are* *natural numbers and define* $A_{i,m_i} = \sum_{s_i=1}^{m_i} a_{i,s_i}$ $(i = 1, 2, \cdots, n)$. *Then*

$$\sum_{m_1=1}^{k_i} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} A_{i,m_i}^{q_i}}{\sum_{i=1}^{n} \alpha_i m_i^{(p_i-1)p}} \le C(k_1, \cdots, k_n) \times \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1)(A_{i,m_i}^{q_i-1} a_{i,m_i})^{p_i} \right\}^{\frac{1}{p_i}}, \tag{2.1}$$

*unless one of* $\{a_{i,m_i}\}$ $(i = 1, 2, \cdots, n)$ *is null, where* $C(k_1, \cdots, k_n) = \frac{\prod_{i=1}^{n} q_i}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} k_i^{\frac{p_i-1}{p_i}}$.

*Proof.* The idea of proof Theorem 2.1 comes from Theorem 1 in Pachpatte[1]. From the hypotheses of Theorem 2.1 and using the following inequality (see[11,12]),

$$\left\{ \sum_{m=1}^{n} z_m \right\}^{\beta} \le \beta \sum_{m=1}^{n} z_m \left\{ \sum_{k=1}^{m} z_k \right\}^{\beta-1}, \tag{2.2}$$

where $\beta \ge 1$ is a constant and $z_m \ge 0$, $(m = 1, 2, \cdots, n)$, it is easy to observe that

$$A_{i,m_i}^{q_i} \le q_i \sum_{s_i=1}^{m_i} A_{i,s_i}^{q_i-1} a_{i,m_i}, \quad m_i = 1, 2, \cdots, k_i, \quad i = 1, 2, \cdots, n. \tag{2.3}$$

From (2.3) and Hölder's inequality, we have

$$\sum_{s_i=1}^{m_i} A_{i,s_i}^{q_i-1} a_{i,m_i} \le m_i^{\frac{p_i-1}{p_i}} \left\{ \sum_{s_i=1}^{m_i} (A_{i,s_i}^{q_i-1} a_{i,s_i})^{p_i} \right\}^{\frac{1}{p_i}}, \tag{2.4}$$

where $m_i = 1, 2, \cdots, k_i$, $i = 1, 2, \cdots, n$. Using the inequality of means[13]

$$\left\{ \prod_{i=1}^{n} x_i^{\omega_i} \right\}^{\frac{1}{\Omega_n}} \le \left\{ \frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i x_i^r \right\}^{\frac{1}{r}} \tag{2.5}$$

for $r > 0$, $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = \Omega_n$. Let $x_i = m_i^{p_i-1}$, $\omega_i = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$ and $r = \sum_{i=1}^{n} \omega_i$, from (2.3)-(2.5), we have

$$\begin{aligned} \prod_{i=1}^{n} A_{i,m_i}^{q_i} &\le \prod_{i=1}^{n} q_i m_i^{\frac{p_i-1}{p_i}} \left\{ \sum_{s_i=1}^{m_i} (A_{i,s_i}^{q_i-1} a_{i,s_i})^{p_i} \right\}^{\frac{1}{p_i}} \\ &\le \frac{1}{p} \left\{ \sum_{i=1}^{n} \frac{m_i^{(p_i-1)p}}{p_i} \right\} \prod_{i=1}^{n} q_i \left\{ \sum_{s_i=1}^{m_i} (A_{i,s_i}^{q_i-1} a_{i,s_i})^{p_i} \right\}^{\frac{1}{p_i}} \\ &= \frac{1}{\sum_{i=1}^{n} \alpha_i} \left\{ \sum_{i=1}^{n} \alpha_i m_i^{(p_i-1)p} \right\} \prod_{i=1}^{n} q_i \left\{ \sum_{s_i=1}^{m_i} (A_{i,s_i}^{q_i-1} a_{i,s_i})^{p_i} \right\}^{\frac{1}{p_i}} \end{aligned} \tag{2.6}$$

for $m_i = 1, 2, \cdots, k_i$. From (2.6), we observe that

$$\frac{\prod_{i=1}^{n} A_{i,m_i}^{q_i}}{\sum_{i=1}^{n} \alpha_i m_i^{(p_i-1)p}} \le \frac{\prod_{i=1}^{n} q_i}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \left\{ \sum_{s_i=1}^{m_i} (A_{i,s_i}^{q_i-1} a_{i,s_i})^{p_i} \right\}^{\frac{1}{p_i}}, \tag{2.7}$$

for $m_i = 1,2,\cdots,k_i$. Taking the sum on both sides of (2.7) first over $m_i$ $(i = 1,2,\cdots,n)$ from 1 to $k_i$ and using Hölder's inequality with indices $p_i$, $p_i/(p_i-1)$ $(i = 1,2,\cdots,n)$ and interchanging the order of summations, we observe that

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}^{q_i}}{\sum_{i=1}^n \alpha_i m_i^{(p_i-1)p}}\\
\le\ & \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{i=1}^{n}\left\{\sum_{s_i=1}^{m_i}(A_{i,s_i}^{q_i-1}a_{i,s_i})^{p_i}\right\}^{\frac{1}{p_i}}\\
=\ & \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left\{\sum_{s_i=1}^{m_i}(A_{i,s_i}^{q_i-1}a_{i,s_i})^{p_i}\right\}^{\frac{1}{p_i}}\right)\\
\le\ & \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^{n}k_i^{\frac{p_i-1}{p_i}}\left\{\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}(A_{i,s_i}^{q_i-1}a_{i,s_i})^{p_i}\right\}^{\frac{1}{p_i}}\\
=\ & \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^{n}k_i^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)(A_{i,m_i}^{q_i-1}a_{i,m_i})^{p_i}\right\}^{\frac{1}{p_i}}.
\end{aligned}$$

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*Remark 1.* In Theorem 2.1, setting $n = 2$, $p_1 = p_2 = 2$, we have (1.1). In Theorem 2.1, setting $p = 1$, we have

$$\sum_{m_1=1}^{k_i}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}^{q_i}}{\sum_{i=1}^n \alpha_i m_i^{p_i-1}} \le C(k_1,\cdots,k_n)\times\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)(A_{i,m_i}^{q_i-1}a_{i,m_i})^{p_i}\right\}^{\frac{1}{p_i}} \tag{2.8}$$

unless one of $\{a_{i,m_i}\}$ $(i = 1,2,\cdots,n)$ is null, where $C(k_1,\cdots,k_n) = \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^{n}k_i^{\frac{p_i-1}{p_i}}$.

*Remark 2.* In Theorem 2.1, setting $q_i = 1$ $(i = 1,2,\cdots,n)$, we have

$$\sum_{m_1=1}^{k_i}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}}{\sum_{i=1}^n \alpha_i m_i^{(p_i-1)p}} \le C_1(k_1,\cdots,k_n)\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)a_{i,m_i}^{p_i}\right\}^{\frac{1}{p_i}} \tag{2.9}$$

unless one of $\{a_{i,m_i}\}$ $(i = 1,2,\cdots,n)$ is null, where $C_1(k_1,\cdots,k_n) = \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^{n}k_i^{\frac{p_i-1}{p_i}}$.

In the following theorem we give the further generalization of the inequality (2.9) obtained in Remark 2.

**Theorem 2.** *Let $p_i > 1$ $(i = 1,2,\cdots,n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1,j\neq i}^n p_j$ $(i = 1,2,\cdots,n)$. Let $\{a_{i,m_i}\}$ $(i = 1,2,\cdots,n)$ be $n$ sequences of nonnegative sequences of real numbers and $\{p_{i,m_i}\}$ $(i = 1,2,\cdots,n)$ be $n$ positive sequences defined for $m_i = 1,2,\cdots,k_i$, where $k_i$ $(i = 1,2,\cdots,n)$ are natural numbers and define $A_{i,m_i} = \sum_{s_i=1}^{m_i} a_{i,s_i}$ and $P_{i,m_i} = \sum_{s_i=1}^{m_i} p_{i,s_i}$ $(i = 1,2,\cdots,n)$. Let $\phi_i$ $(i = 1,2,\cdots,n)$*

*be $n$ real-valued, nonnegative, convex, and submultiplicative functions defined on $R_+ = [0, \infty)$. Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^n \phi_i(A_{i,m_i})}{\sum_{i=1}^n \alpha_i m_i^{(p_i-1)p}} \le M(k_1, \cdots, k_n) \times \prod_{i=1}^n \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left[ p_{i,m_i} \phi_i \left( \frac{a_{i,m_i}}{p_{i,m_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}}, \tag{2.10}$$

*where* $M(k_1, \cdots, k_n) = \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left\{ \sum_{m_i=1}^{k_i} \left[ \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right]^{\frac{p_i}{p_i-1}} \right\}^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses of $\phi_i$ $(i = 1, 2, \cdots, n)$ and by using Jensens inequality and Hölder's inequality, it is easy to observe that

$$\phi_i(A_{i,m_i}) = \phi_i \left( \frac{P_{i,m_i} \sum_{s_i=1}^{m_i} p_{i,s_i} a_{i,s_i}/p_{i,s_i}}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right) \le \phi_i(P_{i,m_i}) \phi_i \left( \frac{\sum_{s_i=1}^{m_i} p_{i,s_i} a_{i,s_i}/p_{i,s_i}}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right)$$
$$\le \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \sum_{s_i=1}^{m_i} p_{i,s_i} \phi_i \left( \frac{a_{i,s_i}}{p_{i,s_i}} \right) \le \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} m_i^{\frac{p_i-1}{p_i}} \left\{ \sum_{s_i=1}^{m_i} \left[ p_{i,s_i} \phi_i \left( \frac{a_{i,s_i}}{p_{i,s_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}}. \tag{2.11}$$

Let $x_i = m_i^{p_i-1}$, $\omega_1 = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$ and $r = \sum_{i=1}^n \omega_i$, from (2.5) and (2.11), we have

$$\prod_{i=1}^n \phi_i(A_{i,m_i}) \le \prod_{i=1}^n \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} m_i^{\frac{p_i-1}{p_i}} \left\{ \sum_{s_i=1}^{m_i} \left[ p_{i,s_i} \phi_i \left( \frac{a_{i,s_i}}{p_{i,s_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}}$$
$$\le \frac{1}{\sum_{i=1}^n \alpha_i} \left\{ \sum_{i=1}^n \alpha_i m_i^{(p_i-1)p} \right\} \prod_{i=1}^n \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \left\{ \sum_{s_i=1}^{m_i} \left[ p_{i,s_i} \phi_i \left( \frac{a_{i,s_i}}{p_{i,s_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \tag{2.12}$$

for $m_i = 1, 2, \cdots, k_i$, $i = 1, 2, \cdots, n$. From (2.12), we observe that

$$\frac{\prod_{i=1}^n \phi_i(A_{i,m_i})}{\sum_{i=1}^n \alpha_i m_i^{(p_i-1)p}} \le \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \left\{ \sum_{s_i=1}^{m_i} \left[ p_{i,s_i} \phi_i \left( \frac{a_{i,s_i}}{p_{i,s_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \tag{2.13}$$

for $m_i = 1, 2, \cdots, k_i$, $i = 1, 2, \cdots, n$. Taking the sum on both sides of (2.13) first over $m_i$ $(i = 1, 2, \cdots, n)$ from 1 to $k_i$ and using Hölder's inequality with indices $p_i$,

$p_i/(p_i-1)$ and interchanging the order of summations, we observe that

$$
\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n\phi_i(A_{i,m_i})}{\sum_{i=1}^n\alpha_i m_i^{(p_i-1)p}}\\
\leq\ & \frac{1}{\sum_{i=1}^n\alpha_i}\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\prod_{i=1}^n\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left\{\sum_{s_i=1}^{m_i}\left[p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}\\
=\ & \frac{1}{\sum_{i=1}^n\alpha_i}\prod_{i=1}^n\left(\sum_{m_i=1}^{k_i}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left\{\sum_{s_i=1}^{m_i}\left[p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}\right)\\
\leq\ & \frac{1}{\sum_{i=1}^n\alpha_i}\prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\left[\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right]^{\frac{p_i}{p_i-1}}\right\}^{\frac{p_i-1}{p_i}}\left\{\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left[p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}\\
=\ & \frac{1}{\sum_{i=1}^n\alpha_i}\prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\left[\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right]^{\frac{p_i}{p_i-1}}\right\}^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left[p_{i,m_i}\phi_i\left(\frac{a_{i,m_i}}{p_{i,m_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}.
\end{aligned}
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*Remark 3.* Let $\omega_i=\frac{1}{p_i}$ $(i=1,2,\cdots,n)$ and $r=\sum\limits_{i=1}^n\omega_i$. If we apply the elementary inequality (2.5) on the right-hand sides of result inequality in Theorem 2.1 and Theorem 2.2, then we get the following inequalities

$$
\sum_{m_1=1}^{k_i}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}^{q_i}}{\sum_{i=1}^n\alpha_i m_i^{(p_i-1)p}}\leq\frac{1}{p}C(k_1,\cdots,k_n)\times\sum_{i=1}^n\frac{1}{p_i}\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)(A_{i,m_i}^{q_i-1}a_{i,m_i})^{p_i}\right\}^p, \tag{2.14}
$$

where $C(k_1,\cdots,k_n)=\frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n\alpha_i}\prod\limits_{i=1}^n k_i^{\frac{p_i-1}{p_i}}$. And

$$
\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n\phi_i(A_{i,m_i})}{\sum_{i=1}^n\alpha_i m_i^{(p_i-1)p}}\leq\frac{1}{p}M(k_1,\cdots,k_n)\times\sum_{i=1}^n\frac{1}{p_i}\left\{\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left[p_{i,m_i}\phi_i\left(\frac{a_{i,m_i}}{p_{i,m_i}}\right)\right]^{p_i}\right\}^p, \tag{2.15}
$$

where $M(k_1,\cdots,k_n)=\frac{1}{\sum_{i=1}^n\alpha_i}\prod\limits_{i=1}^n\left\{\sum\limits_{m_i=1}^{k_i}\left[\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right]^{\frac{p_i}{p_i-1}}\right\}^{\frac{p_i-1}{p_i}}$.

The following theorems deal with slight variants of the inequality (2.11) given in Theorem 2.2.

**Theorem 3.** *Let $p_i>1$ $(i=1,2,\cdots,n)$, $p=\sum_{i=1}^n\frac{1}{p_i}$, $\alpha_i=\prod_{j=1,j\neq i}^n p_j$ $(i=1,2,\cdots,n)$. Let $\{a_{i,m_i}\}$ $(i=1,2,\cdots,n)$ be $n$ sequences of nonnegative sequences of real numbers defined for $m_i=1,2,\cdots,k_i$, where $k_i$ $(i=1,2,\cdots,n)$ are natural* *numbers and define* $A_{i,m_i} = \frac{1}{m_i}\sum_{s_i=1}^{m_i} a_{i,s_i}$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} m_i\phi_i(A_{i,m_i})}{\sum_{i=1}^{n} \alpha_i m_i^{(p_i-1)p}} \le C_1(k_1, \cdots, k_n) \times \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1)\phi_i^{p_i}(a_m) \right\}^{\frac{1}{p_i}}, \tag{2.16}$$

*where* $C_1(k_1, \cdots, k_n) = \frac{1}{\sum_{i=1}^{n}\alpha_i} \prod_{i=1}^{n} k_i^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses and by using Jensens inequality and Hölder's inequality, it is easy to observe that for $i = 1, 2, \cdots, n$,

$$\phi_i(A_{i,m_i}) = \phi_i\left(\frac{1}{m_i}\sum_{s_i=1}^{m_i} a_{i,s_i}\right) \le \frac{1}{m_i}\sum_{s_i=1}^{m_i} \phi_i(a_{i,s_i}) \le \frac{1}{m_i} m_i^{\frac{p_i-1}{p_i}} \left\{\sum_{s_i=1}^{m_i} \phi_i^{p_i}(a_{i,s_i})\right\}^{\frac{p_i-1}{p_i}}. \tag{2.17}$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. ∎

**Theorem 4.** *Let* $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^{n} \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\ne i}^{n} p_j$ $(i = 1, 2, \cdots, n)$. *Let* $\{a_{i,m_i}\}$ $(i = 1, 2, \cdots, n)$ *be* $n$ *sequences of nonnegative sequences of real numbers and* $\{p_{i,m_i}\}$ $(i = 1, 2, \cdots, n)$ *be* $n$ *positive sequences defined for* $m_i = 1, 2, \cdots, k_i$, *where* $k_i$ $(i = 1, 2, \cdots, n)$ *are natural numbers and define* $P_{i,m_i} = \sum_{s_i=1}^{m_i} p_{i,s_i}$ *and* $A_{i,m_i} = \frac{1}{P_{i,m_i}}\sum_{s_i=1}^{m_i} p_{i,s_i}a_{i,s_i}$ $(i = 1, 2, \cdots, n)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} P_{i,m_i}\phi_i(A_{i,m_i})}{\sum_{i=1}^{n} \alpha_i m_i^{(p_i-1)p}} \le C_1(k_1, \cdots, k_n) \times \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1)[p_{i,m_i}\phi_i(a_m)]^{p_i} \right\}^{\frac{1}{p_i}} \tag{2.18}$$

*where* $C_1(k_1, \cdots, k_n) = \frac{1}{\sum_{i=1}^{n}\alpha_i} \prod_{i=1}^{n} k_i^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses and by using Jensens inequality and Hölder's inequality, it is easy to observe that for $i = 1, 2, \cdots, n$,

$$\phi_i(A_{i,m_i}) = \phi_i\left(\frac{1}{P_{i,m_i}}\sum_{s_i=1}^{m_i} p_{i,s_i}a_{i,s_i}\right) \le \frac{1}{P_{i,m_i}}\sum_{s_i=1}^{m_i} p_{i,s_i}\phi_i(a_{i,s_i}) \le \frac{1}{P_{i,m_i}} m_i^{\frac{p_i-1}{p_i}} \left\{\sum_{s_i=1}^{m_i} [p_{i,s_i}\phi_i(a_{i,s_i})]^{p_i}\right\}^{\frac{p_i-1}{p_i}}. \tag{2.19}$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. ∎

Now we also establish the two independent variable versions of the inequalities given in Theorems 2.2.

**Theorem 5.** *Let $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\neq i}^n p_j$ $(i = 1, 2, \cdots, n)$. Let $\{a_{i,m_i,n_i}\}$ $(i = 1, 2, \cdots, n)$ be $n$ sequences of nonnegative sequences of real numbers and $\{p_{i,m_i,n_i}\}$ $(i = 1, 2, \cdots, n)$ be $n$ positive sequences defined for $m_i = 1, 2, \cdots, k_i$, $n_i = 1, 2, \cdots, r_n$, where $k_i$ $(i = 1, 2, \cdots, n)$ are natural numbers and define $A_{i,m_i,n_i} = \sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} a_{i,s_i,t_i}$ and $P_{i,m_i,n_i} = \sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i}$ $(i = 1, 2, \cdots, n)$. Let $\phi_i$ $(i = 1, 2, \cdots, n)$ be $n$ real-valued, nonnegative, convex, and submultiplicative functions defined on $R_+ = [0, \infty)$. Then*

$$\sum_{m_1=1}^{k_1}\sum_{n_1=1}^{r_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_n=1}^{r_n}\frac{\prod_{i=1}^n \phi_i(A_{i,m_i,n_i})}{\sum_{i=1}^n \alpha_i (m_i n_i)^{(p_i-1)p}} \le M(k_1,\cdots,k_n,r_1,\cdots,r_n) \tag{2.20}$$
$$\times\prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\sum_{n_i=1}^{r_i}(k_i-m_i+1)(r_i-n_i+1)\left[p_{i,m_i,n_i}\phi_i\left(\frac{a_{i,m_i,n_i}}{p_{i,m_i,n_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}},$$

*where* $M(k_1,\cdots,k_n,r_1,\cdots,r_n) = \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\sum_{n_i=1}^{r_i}\left[\frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}}\right]^{\frac{p_i}{p_i-1}}\right\}^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses and by using Jensens inequality and Hölder's inequality, it is easy to observe that for $i = 1, 2, \cdots, n$,

$$\begin{aligned}\phi_i(A_{i,m_i,n_i}) &= \phi_i\left(\frac{P_{i,m_i,n_i}\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i} a_{i,s_i,t_i}/p_{i,s_i,t_i}}{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i}}\right)\\ &\le \phi_i(P_{i,m_i,n_i})\phi_i\left(\frac{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i} a_{i,s_i,t_i}/p_{i,s_i,t_i}}{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i}}\right)\\ &\le \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}}\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i}\phi_i\left(\frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}}\right)\\ &\le \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}}(m_i n_i)^{\frac{p_i-1}{p_i}}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}\left[p_{i,s_i,t_i}\phi_i\left(\frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}.\end{aligned} \tag{2.21}$$

Let $x_i = (m_i n_i)^{p_i-1}$, $\omega_1 = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$ and $r = \sum_{i=1}^n \omega_i$, from (2.5) and (2.21), we have

(2.22)

$$\prod_{i=1}^n \phi_i(A_{i,m_i,n_i}) \le \prod_{i=1}^n \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}}(m_i n_i)^{\frac{p_i-1}{p_i}}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}\left[p_{i,s_i,t_i}\phi_i\left(\frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}$$
$$\le \frac{1}{\sum_{i=1}^n \alpha_i}\left\{\sum_{i=1}^n \alpha_i (m_i n_i)^{(p_i-1)p}\right\}\prod_{i=1}^n \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}}\left\{\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i}\left[p_{i,s_i,t_i}\phi_i\left(\frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}}\right)\right]^{p_i}\right\}^{\frac{1}{p_i}}$$

for $m_i = 1, 2, \cdots, k_i$, $n_i = 1, 2, \cdots, n$, $i = 1, 2, \cdots, n$. From (2.22), we observe that

$$\frac{\prod_{i=1}^{n} \phi_i(A_{i,m_i,n_i})}{\sum_{i=1}^{n} \alpha_i (m_i n_i)^{(p_i-1)p}} \le \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}} \times \left\{ \sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} \left[ p_{i,s_i,t_i} \phi_i \left( \frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \tag{2.23}$$

for $m_i = 1, 2, \cdots, k_i$, $n_i = 1, 2, \cdots, n$, $i = 1, 2, \cdots, n$. Taking the sum on both sides of (2.23) first over $m_i$ and $n_i$ $(i = 1, 2, \cdots, n)$ from 1 to $k_i$ and 1 to $r_i$, respectively, and using Hölder's inequality with indices $p_i$, $p_i/(p_i - 1)$ and interchanging the order of summations, we observe that

$$\begin{aligned}
& \sum_{m_1=1}^{k_1} \sum_{n_1=1}^{r_1} \cdots \sum_{m_n=1}^{k_n} \sum_{n_n=1}^{r_n} \frac{\prod_{i=1}^{n} \phi_i(A_{i,m_i,n_i})}{\sum_{i=1}^{n} \alpha_i (m_i n_i)^{(p_i-1)p}} \\
\le\ & \frac{1}{\sum_{i=1}^{n} \alpha_i} \sum_{m_1=1}^{k_1} \sum_{n_1=1}^{r_1} \cdots \sum_{m_n=1}^{k_n} \sum_{n_n=1}^{r_n} \prod_{i=1}^{n} \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}} \left\{ \sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} \left[ p_{i,s_i,t_i} \phi_i \left( \frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \\
=\ & \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \left( \sum_{m_i=1}^{k_i} \sum_{n_i=1}^{r_i} \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}} \left\{ \sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} \left[ p_{i,s_i,t_i} \phi_i \left( \frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \right) \\
\le\ & \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} \sum_{m_i=1}^{r_i} \left[ \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}} \right]^{\frac{p_i}{p_i-1}} \right\}^{\frac{p_i-1}{p_i}} \\
& \times \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} \sum_{n_i=1}^{r_i} \sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} \left[ p_{i,s_i,t_i} \phi_i \left( \frac{a_{i,s_i,t_i}}{p_{i,s_i,t_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}} \\
=\ & \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} \sum_{m_i=1}^{r_i} \left[ \frac{\phi_i(P_{i,m_i,n_i})}{P_{i,m_i,n_i}} \right]^{\frac{p_i}{p_i-1}} \right\}^{\frac{p_i-1}{p_i}} \\
& \times \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} \sum_{n_i=1}^{r_i} (k_i - m_i + 1)(r_i - n_i + 1) \left[ p_{i,m_i,n_i} \phi_i \left( \frac{a_{i,m_i,n_i}}{p_{i,m_i,n_i}} \right) \right]^{p_i} \right\}^{\frac{1}{p_i}}.
\end{aligned}$$

∎

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The following theorems deal with slight variants of the inequality (2.20) given in Theorem 2.5.

**Theorem 6.** *Let $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^{n} \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j \ne i}^{n} p_j$ $(i = 1, 2, \cdots, n)$. Let $\{a_{i,m_i,n_i}\}$ $(i = 1, 2, \cdots, n)$ be $n$ sequences of nonnegative sequences of real numbers defined for $m_i = 1, 2, \cdots, k_i$, $n_i = 1, 2, \cdots, r_i$, where $k_i$ $(i = 1, 2, \cdots, n)$ are natural numbers and define*

$A_{i,m_i,n_i} = \frac{1}{m_i n_i}\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} a_{i,s_i,t_i}$. *Let* $\phi_i$ $(i=1,2,\cdots,n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0,\infty)$. *Then*

(2.24)

$$\sum_{m_1=1}^{k_1}\sum_{n_1=1}^{r_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_n=1}^{r_n}\frac{\prod_{i=1}^n m_i n_i \phi_i(A_{i,m_i,n_i})}{\sum_{i=1}^n \alpha_i (m_i n_i)^{(p_i-1)p}} \le M_1(k_1,\cdots,k_n,r_1,\cdots,r_n) \times \prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\sum_{n_i=1}^{r_i}(k_i-m_i+1)(r_i-n_i+1)\phi_i^{p_i}(a_{i,m_i,n_i})\right\}^{\frac{1}{p_i}},$$

*where* $M_1(k_1,\cdots,k_n,r_1,\cdots,r_n) = \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n (k_i r_i)^{\frac{p_i-1}{p_i}}$.

**Theorem 7.** *Let* $p_i > 1$ $(i=1,2,\cdots,n)$, $p=\sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1,j\neq i}^n p_j$ $(i=1,2,\cdots,n)$. *Let* $\{a_{i,m_i n_i}\}$ $(i=1,2,\cdots,n)$ *be* $n$ *sequences of nonnegative sequences of real numbers and* $\{p_{i,m_i,n_i}\}$ $(i=1,2,\cdots,n)$ *be* $n$ *positive sequences defined for* $m_i=1,2,\cdots,k_i$, $n_i=1,2,\cdots,r_i$, *where* $k_i$ $(i=1,2,\cdots,n)$ *are natural numbers and define* $P_{i,m_i,n_i}=\sum_{s_i=1}^{m_i} p_{i,s_i,t_i}$ *and* $A_{i,m_i,n_i} = \frac{1}{P_{i,m_i,n_i}}\sum_{s_i=1}^{m_i}\sum_{t_i=1}^{n_i} p_{i,s_i,t_i} a_{i,s_i,t_i}$ $(i=1,2,\cdots,n)$. *Let* $\phi_i$ $(i=1,2,\cdots,n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+=[0,\infty)$. *Then*

(2.25)

$$\sum_{m_1=1}^{k_1}\sum_{n_1=1}^{r_1}\cdots\sum_{m_n=1}^{k_n}\sum_{n_n=1}^{r_n}\frac{\prod_{i=1}^n P_{i,m_i,n_i}\phi_i(A_{i,m_i,n_i})}{\sum_{i=1}^n \alpha_i (m_i n_i)^{(p_i-1)p}} \le M_1(k_1,\cdots,k_n,r_1,\cdots,r_n) \times \prod_{i=1}^n\left\{\sum_{m_i=1}^{k_i}\sum_{n_i=1}^{r_i}(k_i-m_i+1)(r_i-n_i+1)[p_{i,m_i,n_i}\phi_i(a_{i,m_i,n_i})]^{p_i}\right\}^{\frac{1}{p_i}}$$

*where* $M_1(k_1,\cdots,k_n,r_1,\cdots,r_n) = \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n (k_i r_i)^{\frac{p_i-1}{p_i}}$.

The proof of Theorem 2.6 and Theorem 2.7 can be completed by following the same steps as in the proofs of Theorems 2.5 with suitable changes and hence we omit the details.

## 3. Integral analogues

Now we give the integral analogues of the inequalities given in Theorems 2.1-2.7. An integral analogue of Theorem 2.1 is given in the following theorem.

**Theorem 8.** *Let* $p_i > 1$ $(i=1,2,\cdots,n)$, $p=\sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i=\prod_{j=1,j\neq i}^n p_j$ $(i=1,2,\cdots,n)$. *Let* $f_i(\tau_i)\ge 0$ *for* $\tau_i\in(0,x_i)$, *where* $x_i$ $(i=1,2,\cdots,n)$ *are positive real numbers, define* $F(s_i)=\int_0^{s_i} f_i(\tau_i)d\tau_i$ $(i=1,2,\cdots,n)$ *for* $s_i\in(0,x_i)$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i^{q_i}(s_i)}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n \le D(x_1,\cdots,x_n) \times\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)(F_i^{q_i-1}(s_i)f_i(s_i))^{p_i}ds_i\right\}^{\frac{1}{p_i}}, \tag{3.1}$$

*unless one of* $f_i(\tau_i)$ $(i = 1, 2, \cdots, n)$ *is 0, where* $D(x_1, \cdots, x_n) = \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses of Theorem 3.1, it is easy to observe that

$$F_i^{q_i}(s_i) = q_i \int_0^{s_i} F_i^{q_i-1}(\tau_i) f(\tau_i) d\tau_i, \quad s_i \in (0, x_i), \quad i = 1, 2, \cdots, n. \tag{3.2}$$

From (3.2) and Hölder's inequality, we have

$$\int_0^{x_i} F_i^{q_i-1}(\tau_i) f(\tau_i) d\tau_i \leq s_i^{\frac{p_i-1}{p_i}} \left\{ \int_0^{s_i} (F_i^{q_i-1}(\tau_i) f(\tau_i))^{p_i} d\tau_i \right\}^{\frac{1}{p_i}}, \tag{3.3}$$

where $s_i \in (0, s_i)$, $i = 1, 2, \cdots, n$.

Let $x_i = s_i^{p_i-1}$, $\omega_1 = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$ and $r = \sum_{i=1}^n \omega_i$, from (2.5) and (3.3), we observe that

$$\begin{aligned} \prod_{i=1}^n F_i^{q_i}(s_i) &\leq \prod_{i=1}^n q_i s_i^{\frac{p_i-1}{p_i}} \left\{ \int_0^{s_i} (F_i^{q_i-1}(\tau_i) f(\tau_i))^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \\ &\leq \frac{1}{\sum_{i=1}^n \alpha_i} \left\{ \sum_{i=1}^n \alpha_i s_i^{(p_i-1)p} \right\} \prod_{i=1}^n q_i \left\{ \int_0^{s_i} (F_i^{q_i-1}(\tau_i) f(\tau_i))^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \end{aligned} \tag{3.4}$$

for $s_i \in (0, x_i)$, $i = 1, 2, \cdots, n$. From (3.4), we observe that

$$\frac{\prod_{i=1}^n F_i^{q_i}(s_i)}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}} \leq \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n q_i \left\{ \int_0^{s_i} (F_i^{q_i-1}(\tau_i) f(\tau_i))^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \tag{3.5}$$

for $s_i \in (0, x_i)$, $i = 1, 2, \cdots, n$. Taking the integral on both sides of (3.5) over $s_i$ $(i = 1, 2, \cdots, n)$ from 0 to $x_i$ and using Hölder's inequality with indices $p_i$, $p_i/(p_i - 1)$ and interchanging the order of integrals, we observe that

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i^{q_i}(s_i)}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n$$

$$\leq \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\int_0^{x_1}\cdots\int_0^{x_n}\prod_{i=1}^n\left\{\int_0^{s_i}(F_i^{q_i-1}(\tau_i)f(\tau_i))^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_1\cdots ds_n$$

$$= \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left(\int_0^{x_i}\left\{\int_0^{s_i}(F_i^{q_i-1}(\tau_i)f(\tau_i))^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_i\right)$$

$$\leq \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}\left\{\int_0^{x_i}\int_0^{s_i}(F_i^{q_i-1}(\tau_i)f(\tau_i))^{p_i}d\tau_i ds_i\right\}^{\frac{1}{p_i}}$$

$$= \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)(F_i^{q_i-1}(s_i)f(s_i))^{p_i}ds_i\right\}^{\frac{1}{p_i}}.$$

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*Remark 4.* In Theorem 2.1, setting $n = 2$, $p_1 = p_2 = 2$, we have (1.2). In Theorem 2.1, setting $p = 1$, we have

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i^{q_i}(s_i)}{\sum_{i=1}^n \alpha_i s_i^{p_i-1}}ds_1\cdots ds_n \leq D(x_1,\cdots,x_n) \times\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)(F_i^{q_i-1}(s_i)f_i(s_i))^{p_i}ds_i\right\}^{\frac{1}{p_i}}, \tag{3.6}$$

unless one of $f_i(\tau_i)$ $(i = 1, 2, \cdots, n)$ is 0, where $D(x_1,\cdots,x_n) = \frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$.

*Remark 5.* In Theorem 2.1, setting $q_i = 1$ $(i = 1, 2, \cdots, n)$, we have

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i(s_i)}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n \leq D_1(x_1,\cdots,x_n) \times\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)f_i^{p_i}(s_i)ds_i\right\}^{\frac{1}{p_i}}, \tag{3.7}$$

unless one of $f_i(\tau_i)$ $(i = 1, 2, \cdots, n)$ is 0, where $D_1(x_1,\cdots,x_n) = \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$.

In the following theorem we give the further generalization of the inequality (3.7) obtained in Remark 5.

**Theorem 9.** *Let $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1,j\neq i}^n p_j$ $(i = 1, 2, \cdots, n)$. Let $f_i(\tau_i) \geq 0$ and $p_i(\tau_i) > 0$ for $\tau_i \in (0, x_i)$, where $x_i$ $(i = 1, 2, \cdots, n)$*

*are positive real numbers, define* $F_i(s_i) = \int_0^{s_i} f_i(\tau_i)d\tau_i$ *and* $P_i(s_i) = \int_0^{s_i} p_i(\tau_i)d\tau_i$ $(i = 1, 2, \cdots, n)$ *for* $s_i \in (0, x_i)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex, and submultiplicative functions defined on* $R_+ = [0, \infty)$. *Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}} ds_1 \cdots ds_n \le L(x_1, \cdots, x_n) \times \prod_{i=1}^n \left\{ \int_0^{x_i} (x_i - s_i) \left[ p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right]^{p_i} ds_i \right\}^{\frac{1}{p_i}}, \tag{3.8}$$

*where* $L(x_1, \cdots, x_n) = \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left\{ \int_0^{x_i} \left[\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right]^{\frac{p_i}{p_i-1}} ds_i \right\}^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses of Theorem 3.2 and by using Jensens inequality and Hölder's inequality, it is easy to see that $for i = 1, 2, \cdots, n$,

$$\begin{aligned} \phi_i(F_i(s_i)) &= \phi_i\left(\frac{P_i(s_i)\int_0^{s_i} p_i(\tau_i)(f_i(\tau_i)/p_i(\tau_i))d\tau_i}{\int_0^{s_i} p_i(\tau_i)d\tau_i}\right) \\ &\le \phi_i(P_i(s_i))\phi_i\left(\frac{\int_0^{s_i} p_i(\tau_i)(f_i(\tau_i)/p_i(\tau_i))d\tau_i}{\int_0^{s_i} p(\tau_i)d\tau_i}\right) \\ &\le \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \int_0^{s_i} p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right) d\tau_i \\ &\le \frac{\phi_i(P_i(s_i))}{P_i(s_i)} s_i^{\frac{p_i-1}{p_i}} \left\{ \int_0^{s_i} \left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i} d\tau_i \right\}^{\frac{1}{p_i}}. \end{aligned} \tag{3.9}$$

Let $x_1 = s_i^{\alpha-1}$, $\omega_i = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$, $r = \sum_{i=1}^n \omega_i$, from (3.9) and (2.7), we observe that

$$\begin{aligned} \prod_{i=1}^n \phi_i(F_i(s_i)) &\le \prod_{i=1}^n \frac{\phi_i(P_i(s_i))}{P_i(s_i)} s_i^{\frac{p_i-1}{p_i}} \left\{ \int_0^{s_i} \left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \\ &\le \frac{1}{\sum_{i=1}^n \alpha_i} \left\{ \sum_{i=1}^n \alpha_i s_i^{(p_i-1)p} \right\} \prod_{i=1}^n \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \left\{ \int_0^{s_i} \left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \end{aligned} \tag{3.10}$$

for $s_i \in (0, x_i)$, $i = 1, 2, \cdots, n$. From (3.10), we observe that

$$\frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}} \le \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \frac{\phi_i(P_i(s_i)}{P_i(s_i)} \left\{ \int_0^{s_i} \left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i} d\tau_i \right\}^{\frac{1}{p_i}} \tag{3.11}$$

for $s_i \in (0, x_i)$, $i = 1, 2, \cdots, n$. Taking the integral on both sides of (3.11) over $s_i$ $(i = 1, 2, \cdots, n)$ from 0 to $x_i$ and using Hölder's inequality with indices $p_i$,

$p_i/(p_i-1)$ and interchanging the order of integrals, we observe that

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n\\
\le\ & \frac{1}{\sum_{i=1}^n \alpha_i}\int_0^{x_1}\cdots\int_0^{x_n}\prod_{i=1}^n \frac{\phi_i(P_i(s_i)}{P_i(s_i)}\left\{\int_0^{s_i}\left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_1\cdots ds_n\\
=\ & \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left(\int_0^{x_i}\frac{\phi_i(P_i(s_i)}{P_i(s_i)}\left\{\int_0^{s_i}\left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i}d\tau_i\right\}^{\frac{1}{p_i}}ds_i\right)\\
\le\ & \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left\{\int_0^{x_i}\left[\frac{\phi_i(P_i(s_i)}{P_i(s_i)}\right]^{\frac{p_i}{p_i-1}}ds_i\right\}^{\frac{p_i-1}{p_i}}\left\{\int_0^{x_i}\int_0^{s_i}\left[p_i(\tau_i)\phi_i\left(\frac{f_i(\tau_i)}{p_i(\tau_i)}\right)\right]^{p_i}d\tau_i ds_i\right\}^{\frac{1}{p_i}}\\
=\ & \frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left\{\int_0^{x_i}\left[\frac{\phi_i(P_i(s_i)}{P_i(s_i)}\right]^{\frac{p_i}{p_i-1}}ds_i\right\}^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left\{\int_0^{x_i}(x_i-s_i)\left[p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right]^{p_i}ds_i\right\}^{\frac{1}{p_i}}.
\end{aligned}$$

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*Remark 6.* If we apply the elementary inequality (2.5) on the right-hand sides of result inequality in Theorem 3.1 and Theorem 3.2, then we get the following inequalities

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n F_i^{q_i}(s_i)}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n \le \frac{1}{p}D(x_1,\cdots,x_n)\times\sum_{i=1}^n\frac{1}{p_i}\left\{\int_0^{x_i}(x_i-s_i)(F_i^{q_i-1}(s_i)f_i(s_i))^{p_i}ds_i\right\}^p, \tag{3.12}$$

where $D(x_1,\cdots,x_n)=\frac{\prod_{i=1}^n q_i}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$. And

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}}ds_1\cdots ds_n \le \frac{1}{p}L(x_1,\cdots,x_n)\times\sum_{i=1}^n\frac{1}{p_i}\left\{\int_0^{x_i}(x_i-s_i)\left[p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right]^{p_i}ds_i\right\}^p, \tag{3.13}$$

where $L(x_1,\cdots,x_n)=\frac{1}{\sum_{i=1}^n \alpha_i}\prod_{i=1}^n\left\{\int_0^{x_i}\left[\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right]^{\frac{p_i}{p_i-1}}ds_i\right\}^{\frac{1}{p_i}}$.

The following theorems deal with slight variants of (3.8) given in Theorem 3.2.

**Theorem 10.** *Let $p_i>1$ $(i=1,2,\cdots,n)$, $p=\sum_{i=1}^n\frac{1}{p_i}$, $\alpha_i=\prod_{j=1,j\neq i}^n p_j$ $(i=1,2,\cdots,n)$. Let $f_i(\tau_i)\ge 0$ for $\tau_i\in(0,x_i)$, where $x_i$ $(i=1,2,\cdots,n)$ are positive* *real numbers, define* $F_i(s_i) = \frac{1}{s_i}\int_0^{s_i} f_i(\tau_i)d\tau_i$ $(i = 1, 2, \cdots, n)$ *for* $s_i \in (0, x_i)$. *Let Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n s_i\phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}} ds_1\cdots ds_n \le D_1(x_1,\cdots,x_n) \times \prod_{i=1}^n \left\{ \int_0^{x_i} (x_i - s_i)\phi_i^{p_i}(f_i(s_i))ds_i \right\}^{\frac{1}{p_i}}, \tag{3.14}$$

*where* $D_1(x_1, \cdots, x_n) = \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$.

**Theorem 11.** *Let* $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\neq i}^n p_j$ $(i = 1, 2, \cdots, n)$. *Let* $f_i(\tau_i) \ge 0$ *and* $p_i(\tau_i) > 0$ *for* $\tau_i \in (0, x_i)$, *where* $x_i$ $(i = 1, 2, \cdots, n)$ *are positive real numbers, define* $P_i(s_i) = \int_0^{s_i} p_i(\tau_i)d\tau_i$ *and* $F_i(s_i) = \frac{1}{P_i(s_i)}\int_0^{s_i} f_i(\tau_i)d\tau_i$ $(i = 1, 2, \cdots, n)$ *for* $s_i \in (0, x_i)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n P_i(s_i)\phi_i(F_i(s_i))}{\sum_{i=1}^n \alpha_i s_i^{(p_i-1)p}} ds_1\cdots ds_n \le D_1(x_1,\cdots,x_n) \times \prod_{i=1}^n \left\{ \int_0^{x_i} (x_i - s_i)\left[p_i(s_i)\phi_i\left(f_i(s_i)\right)\right]^{p_i} ds \right\}^{\frac{1}{p_i}}, \tag{3.15}$$

*where* $D_1(x_1, \cdots, x_n) = \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}}$.

The proofs of Theorems 3.3 and 3.4 are similar to the proof of Theorem 3.2 and close to the proofs of Theorems 2.3 and 2.4. Hence, we leave out the details.

Now we also establish the two independent variable versions of the inequalities given in Theorems 3.2.

**Theorem 12.** *Let* $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^n \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\neq i}^n p_j$ $(i = 1, 2, \cdots, n)$. *Let* $f_i(s_i, t_i)$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued continuous functions and* $p_i(s_i, t_i)$ $(i = 1, 2, \cdots, n)$ *be* $n$ *positive continuous functions defined for* $(s_i, t_i) \in [0, x_i) \times [0, y_i)$, *where* $x_i \in (0, \infty)$, $y_i \in (0, \infty)$ *and define* $F_i(s_i, t_i) = \int_0^{s_i}\int_0^{t_i} f_i(\sigma_i, \tau_i)d\sigma_i d\tau_i$ *and* $P_i(s_i, t_i) = \int_0^{s_i}\int_0^{t_i} p_i(\sigma_i, \tau_i)d\sigma_i d\tau_i$ $(i = 1, 2, \cdots, n)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex, and submultiplicative* *functions defined on* $R_+ = [0, \infty)$. *Then*

(3.16)
$$\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n \phi_i(F_i(s_i,t_i))}{\sum_{i=1}^n \alpha_i(s_it_i)^{(p_i-1)p}}ds_1dt_1\cdots ds_ndt_n \le K(x_1,\cdots,x_n,y_1,\cdots,y_n)$$
$$\times\prod_{i=1}^n\left\{\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left[p_i(s_i,t_i)\phi_i\left(\frac{f_i(s_i,t_i)}{p_i(s_i,t_i)}\right)\right]^{p_i}ds_idt_i\right\}^{\frac{1}{p_i}},$$

*where* $K(x_1,\cdots,x_n,y_1,\cdots,y_n) = \frac{1}{\sum_{i=1}^n\alpha_i}\prod_{i=1}^n\left\{\int_0^{x_i}\int_0^{y_i}\left[\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right]^{\frac{p_i}{p_i-1}}ds_idt_i\right\}^{\frac{p_i-1}{p_i}}$.

*Proof.* From the hypotheses and by using Jensens inequality and Hölder's inequality, it is easy to observe that for $i = 1, 2, \cdots, n$,

$$\phi_i(F_i(s_i,t_i)) = \phi_i\left(\frac{P_i(s_i,t_i)\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)f_i(\sigma_i,\tau_i)/p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}\right) \tag{3.17}$$
$$\le \phi_i(P_i(s_i,t_i))\phi_i\left(\frac{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)f_i(\sigma_i,\tau_i)/p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}\right)$$
$$\le \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)d\sigma_id\tau_i$$
$$\le \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}(s_it_i)^{\frac{p_i-1}{p_i}}\left\{\int_0^{s_i}\int_0^{t_i}\left[p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i}d\sigma_id\tau_i\right\}^{\frac{1}{p_i}}.$$

Let $x_i = (s_it_i)^{p_i-1}$, $\omega_1 = \frac{1}{p_i}$ $(i = 1, 2, \cdots, n)$ and $r = \sum_{i=1}^n \omega_i$, from (2.5) and (3.17), we have

$$\prod_{i=1}^n\phi_i(F_i(s_i,t_i)) \le \prod_{i=1}^n\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}(s_it_i)^{\frac{p_i-1}{p_i}} \tag{3.18}$$
$$\times\left\{\int_0^{s_i}\int_0^{t_i}\left[p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i}d\sigma_id\tau_i\right\}^{\frac{1}{p_i}}$$
$$\le \frac{1}{\sum_{i=1}^n\alpha_i}\left\{\sum_{i=1}^n\alpha_i(s_it_i)^{(p_i-1)p}\right\}\prod_{i=1}^n\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}$$
$$\times\left\{\int_0^{s_i}\int_0^{t_i}\left[p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i}d\sigma_id\tau_i\right\}^{\frac{1}{p_i}}$$

for $s_i \in (0, x_i)$ and $t_i \in (0, y_i)$ $i = 1, 2, \cdots, n$. From (3.18), we observe that

$$\frac{\prod_{i=1}^n \phi_i(F_i(s_i,t_i))}{\sum_{i=1}^n \alpha_i (s_i t_i)^{(p_i-1)p}} \leq \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)} \times \left\{ \int_0^{s_i}\int_0^{t_i} \left[ p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i} d\sigma_i d\tau_i \right\}^{\frac{1}{p_i}} \tag{3.19}$$

for $s_i \in (0, x_i)$ and $t_i \in (0, y_i)$ $i = 1, 2, \cdots, n$. Taking the integral on both sides of (3.19) $s_i$ and $t_i$ $(i = 1, 2, \cdots, n)$ from 0 to $x_i$ and 0 to $y_i$, respectively, and using Hölder's inequality with indices $p_i$, $p_i/(p_i - 1)$ and interchanging the order of integrals, we observe that

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n} \frac{\prod_{i=1}^n \phi_i(F_i(s_i,t_i))}{\sum_{i=1}^n \alpha_i (s_i t_i)^{(p_i-1)p}} ds_1 dt_1 \cdots ds_n dt_n \\
\leq\ & \frac{1}{\sum_{i=1}^n \alpha_i} \int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n} \prod_{i=1}^n \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)} \\
&\times \left\{ \int_0^{s_i}\int_0^{t_i} \left[ p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i} d\sigma_i d\tau_i \right\}^{\frac{1}{p_i}} ds_1 dt_1 \cdots ds_n dt_n \\
=\ & \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left( \int_0^{x_i}\int_0^{y_i} \frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)} \right. \\
&\left. \times \left\{ \int_0^{s_i}\int_0^{t_i} \left[ p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i} d\sigma_i d\tau_i \right\}^{\frac{1}{p_i}} ds_i dt_i \right) \\
\leq\ & \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left\{ \int_0^{x_i}\int_0^{y_i} \left[\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right]^{\frac{p_i}{p_i-1}} ds_i dt_i \right\}^{\frac{p_i-1}{p_i}} \\
&\times \prod_{i=1}^n \left\{ \int_0^{x_i}\int_0^{y_i}\int_0^{s_i}\int_0^{t_i} \left[ p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right]^{p_i} d\sigma_i d\tau_i ds_i dt_i \right\}^{\frac{1}{p_i}} \\
=\ & \frac{1}{\sum_{i=1}^n \alpha_i} \prod_{i=1}^n \left\{ \int_0^{x_i}\int_0^{y_i} \left[\frac{\phi_i(P_i(s_i,t_i))}{P_i(s_i,t_i)}\right]^{\frac{p_i}{p_i-1}} ds_i dt_i \right\}^{\frac{p_i-1}{p_i}} \\
&\times \prod_{i=1}^n \left\{ \int_0^{x_i}\int_0^{y_i} (x_i - s_i)(y_i - t_i) \left[ p_i(s_i,t_i)\phi_i\left(\frac{f_i(s_i,t_i)}{p_i(s_i,t_i)}\right)\right]^{p_i} ds_i dt_i \right\}^{\frac{1}{p_i}}.
\end{aligned}$$

∎

The following theorems deal with slight variants of the inequality (3.16) given in Theorem 3.5.

**Theorem 13.** *Let* $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^{n} \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\neq i}^{n} p_j$ $(i = 1, 2, \cdots, n)$. *Let* $f_i(s_i, t_i)$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued continuous functions defined for* $(s_i, t_i) \in [0, x_i) \times [0, y_i)$, *where* $x_i \in (0, \infty)$, $y_i \in (0, \infty)$ *and define* $F_i(s_i, t_i) = \frac{1}{s_i t_i} \int_0^{s_i} \int_0^{t_i} f_i(\sigma_i, \tau_i) d\sigma_i d\tau_i$ $(i = 1, 2, \cdots, n)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

(3.20)
$$\int_0^{x_1}\int_0^{y_1} \cdots \int_0^{x_n}\int_0^{y_n} \frac{\prod_{i=1}^{n} s_i t_i \phi_i(F_i(s_i, t_i))}{\sum_{i=1}^{n} \alpha_i (s_i t_i)^{(p_i - 1)p}} ds_1 dt_1 \cdots ds_n dt_n \leq K_1(x_1, \cdots, x_n, y_1, \cdots, y_n)$$
$$\times \prod_{i=1}^{n} \left\{ \int_0^{x_i}\int_0^{y_i} (x_i - s_i)(y_i - t_i) \phi_i^{p_i}(f_i(s_i, t_i)) ds_i dt_i \right\}^{\frac{1}{p_i}},$$

*where* $K_1(x_1, \cdots, x_n, y_1, \cdots, y_n) = \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} (x_i y_i)^{\frac{p_i - 1}{p_i}}$.

**Theorem 14.** *Let* $p_i > 1$ $(i = 1, 2, \cdots, n)$, $p = \sum_{i=1}^{n} \frac{1}{p_i}$, $\alpha_i = \prod_{j=1, j\neq i}^{n} p_j$ $(i = 1, 2, \cdots, n)$. *Let* $f_i(s_i, t_i)$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued continuous functions and* $p_i(s_i, t_i)$ $(i = 1, 2, \cdots, n)$ *be* $n$ *positive continuous functions defined for* $(s_i, t_i) \in [0, x_i) \times [0, y_i)$, *where* $x_i \in (0, \infty)$, $y_i \in (0, \infty)$ *and define* $P_i(s_i, t_i) = \int_0^{s_i} \int_0^{t_i} p_i(\sigma_i, \tau_i) d\sigma_i d\tau_i$ *and* $F_i(s_i, t_i) = \frac{1}{P_i(s_i, t_i)} \int_0^{s_i} \int_0^{t_i} f_i(\sigma_i, \tau_i) d\sigma_i d\tau_i$ $(i = 1, 2, \cdots, n)$. *Let* $\phi_i$ $(i = 1, 2, \cdots, n)$ *be* $n$ *real-valued, nonnegative, convex functions defined on* $R_+ = [0, \infty)$. *Then*

$$\int_0^{x_1}\int_0^{y_1} \cdots \int_0^{x_n}\int_0^{y_n} \frac{\prod_{i=1}^{n} P_i(s_i, t_i) \phi_i(F_i(s_i, t_i))}{\sum_{i=1}^{n} \alpha_i (s_i t_i)^{(p_i - 1)p}} ds_1 dt_1 \cdots ds_n dt_n \tag{3.21}$$
$$\leq K_1(x_1, \cdots, x_n, y_1, \cdots, y_n)$$
$$\times \prod_{i=1}^{n} \left\{ \int_0^{x_i}\int_0^{y_i} (x_i - s_i)(y_i - t_i) [p_i(s_i, t_i) \phi_i(f_i(s_i, t_i))]^{p_i} ds_i dt_i \right\}^{\frac{1}{p_i}}$$

*where* $K_1(x_1, \cdots, x_n, y_1, \cdots, y_n) = \frac{1}{\sum_{i=1}^{n} \alpha_i} \prod_{i=1}^{n} (x_i y_i)^{\frac{p_i - 1}{p_i}}$.

The proof of Theorem 3.6 and Theorem 3.7 can be completed by following the same steps as in the proofs of Theorems 3.5 with suitable changes and hence we omit the details.

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