SOME NEW INEQUALITIES SIMILAR TO HILBERT-PACHPATTE'S INEQUALITIES WITH TWO INDEPENDENT VARIABLE

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Abstract. Some new inequalities similar to Hilbert-Pachpatte discrete inequalities with two independent variable are established in this paper. The integral analogues of the main results are also given.

1. Introduction

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_m, $b_n \geq 0$. If $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{m=1}^{\infty} a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

Inequality (1.1) is the well known Hardy-Hilbert's inequality[1]. It is well known that Hardy-Hilbert's inequalities play a dominant role in analysis, so the literature on such inequalities and their applications is vast[2-6]. By defining the operator ∇, Pachpatte[7] gives new estimates on inequalities of this type.

In order to narrate conveniently, we give first notations and definitions[7,8]. Let R denote the set of real numbers, $N = \{1, 2, \cdots\}$, $N_0 = \{0, 1, 2, \cdots\}$, $N_\alpha = \{1, 2, \cdots, \alpha\}$, $\alpha \in N$. Define the operator ∇ by $\nabla u(t) = u(t) - u(t-1)$ for any function u defined on N_0. And define the operators $\nabla_1 v(s,t) = v(s,t) - v(s-1,t)$, $\nabla_2 v(s,t) = v(s,t) - v(s,t-1)$ and $\nabla_2\nabla_1 v(s,t) = \nabla_2(\nabla_1 v(s,t)) = \nabla_1(\nabla_2 v(s,t))$ for any function $v(s,t) : N_0 \times N_0 \to R$.

Now we repeat the result of [7] as follows.

Theorem 1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $a(s) : N_m \to R$, $b(t) : N_n \to R$, and $a(0) = b(0) = 0$. Then*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{|a(s)||b(t)|}{qs^{p-1} + pt^{q-1}} \leq M(p,q,m,n)\left\{\sum_{s=1}^{m}(m-s+1)|\nabla a(s)|^p\right\}^{\frac{1}{p}} \times \left\{\sum_{t=1}^{n}(n-t+1)|\nabla b(t)|^q\right\}^{\frac{1}{q}} \tag{1.2}$$

for $m, n \in N$, where $M(p,q,m,n) = \frac{1}{pq} m^{(p-1)/p} n^{(q-1)/q}$, for $m, n \in N$.

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Inequality (1.2) is the well known Hilbert-Pachpatte's inequality. Subsequently, a series of results are given[13-15].

Lv[8] gives a inequality similar to (1.2) as follows.

Theorem 2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $a(m)$ and $b(n)$ be two sequences of real numbers where $m, n \in N_0$, and and $a(0) = b(0) = 0$. If $0 < \sum_{m=1}^{\infty} \sum_{\tau=1}^{m} |\nabla a(\tau)|^p < \infty$ and $0 < \sum_{n=1}^{\infty} \sum_{\delta=1}^{n} |\nabla b(\delta)|^q < \infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|a(m)||b(n)|}{(qm^{p-1} + pn^{q-1})(m+n)} \leq \frac{\pi}{pq\sin(\pi/p)} \left\{\sum_{m=1}^{\infty}\sum_{k=1}^{m} |\nabla a(k)|^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^{\infty}\sum_{r=1}^{n} |\nabla b(r)|^q\right\}^{\frac{1}{q}}. \tag{1.3}$$

The above inequality (1.3) takes the form of strict inequality. The following inequality with two independent variable similar to inequality (1.2) is obtained by Pachpatte[7].

Theorem 3. *Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. Let $a(s,t) : N_x \times N_y \to R$, $b(k,r) : N_z \times N_w \to R$, and $a(0,t) = b(0,t) = a(s,0) = b(s,0) = 0$. Then*

$$\sum_{s=1}^{x}\sum_{t=1}^{y}\sum_{k=1}^{z}\sum_{r=1}^{w} \frac{|a(s,t)||b(k,r)|}{q(st)^{p-1} + p(kr)^{q-1}} \leq L(p,q,x,y,z,w) \times \left\{\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1)|\nabla_2\nabla_1 a(s,t)|^p\right\}^{\frac{1}{p}} \times \left\{\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1)|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}} \tag{1.4}$$

for $x, y, z, w \in N$, where $L(p,q,x,y,z,w) = \frac{1}{pq}(xy)^{(p-1)/p}(zw)^{(q-1)/q}$, for $x, y, z, w \in N$.

The purpose of this paper is to build some new inequalities with two independent variable similar to inequality (1.3) and (1.4). At last, the integral analogues of the main results are also given.

2. Some lemmas

First we introduce some lemmas in the course of proof of the main results. Let $B(\bullet, *)$ denote beta function in this paper.

Lemma 2.1.[9] *Let $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $2-\min\{p,q\} < \lambda$. Define the weight function $\omega_1(q,m)$ and $\bar{\omega}_1(q,x)$ as*

$$\omega_1(q,m) := \sum_{n=1}^{\infty} \frac{1}{(m+n)^{\lambda}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}, m \in \{1, 2\cdots\}$$

and

$$\bar{\omega}_1(q,x) := \int_0^{\infty} \frac{1}{(x+y)^{\lambda}} \left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy, x \in (0,\infty).$$

Then

(1) $\bar{\omega}_1(q,x) = B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right) x^{1-\lambda}$.

(2) If $2 - \min\{p,q\} < \lambda \le 2 + \min\{p,q\}$, *then* $\omega_1(q,m) < \bar{\omega}_1(q,m)$.

Lemma 2.2.[8,10] *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p,q\} < \lambda$. *Define the weight function* $\omega_2(q,m)$ *and* $\bar{\omega}_2(q,x)$ *as*

$$\omega_2(q,m) := \sum_{n=1}^{\infty} \frac{1}{m^\lambda + n^\lambda} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}}, m \in \{1, 2 \cdots\}$$

and

$$\bar{\omega}_2(q,x) := \int_0^\infty \frac{1}{x^\lambda + y^\lambda} \left(\frac{x}{y}\right)^{\frac{2-\lambda}{q}} dy, x \in (0,\infty).$$

Then

(1) $\bar{\omega}_2(q,x) = \frac{1}{\lambda} B\left(\frac{q+\lambda-2}{q\lambda}, \frac{p+\lambda-2}{p\lambda}\right) x^{1-\lambda}$.

(2) If $2 - \min\{p,q\} < \lambda \le 2$, *then* $\omega_2(q,m) < \bar{\omega}_2(q,m)$.

Lemma 2.3.[11] *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda$. *Define the weight function* $\omega_3(q,m)$ *and* $\bar{\omega}_3(q,x)$ *as*

$$\omega_3(q,m) := \sum_{n=1}^{\infty} \frac{\ln\left(\frac{m}{n}\right)}{m^\lambda - n^\lambda} \left(\frac{m^\lambda}{n^\lambda}\right)^{\frac{1}{q}} \frac{m^{(p-1)(1-\lambda)}}{n^{1-\lambda}}, \quad m \in \{1, 2 \cdots\}$$

and

$$\bar{\omega}_3(q,x) := \int_0^\infty \frac{\ln\left(\frac{x}{y}\right)}{x^\lambda - y^\lambda} \left(\frac{x^\lambda}{y^\lambda}\right)^{\frac{1}{q}} \frac{x^{(p-1)(1-\lambda)}}{y^{1-\lambda}} dy, \quad x \in (0,\infty).$$

Then

(1) $\bar{\omega}_3(q,x) = \left[\frac{\pi}{\lambda \sin(\pi/p)}\right]^2 x^{(p-1)(1-\lambda)}$.

(2) If $0 < \lambda \le \min\{p,q\}$, *then* $\omega_3(q,m) < \bar{\omega}_3(q,m)$.

Lemma 2.4.[12] *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda_1, \lambda_2$. *Define the weight function* $\omega_4(q,m)$ *and* $\bar{\omega}_4(q,x)$ *as*

$$\omega_4(q,m) := \sum_{n=1}^{\infty} \frac{1}{m^{\lambda_1} + n^{\lambda_2}} \left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)^{\frac{1}{q}} \frac{m^{(p-1)(1-\lambda_1)}}{n^{1-\lambda_2}}, \quad m \in \{1, 2 \cdots\}$$

and

$$\bar{\omega}_4(q,x) := \int_0^\infty \frac{1}{x^{\lambda_1} + y^{\lambda_2}} \left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{1}{q}} \frac{x^{(p-1)(1-\lambda_1)}}{y^{1-\lambda_2}} dy, \quad x \in (0,\infty).$$

Then

(1) $\bar{\omega}_4(q,x) = \frac{\pi}{\lambda_2 \sin(\pi/p)} x^{(p-1)(1-\lambda_1)}$.

(2) If $0 < \lambda_1 < q$, $0 < \lambda_2 < p$, *then* $\omega_4(q,m) < \bar{\omega}_4(q,m)$.

3. Discrete case

Theorem 4. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $a(s,t) : N_0 \times N_0 \to R$, $b(k,r) : N_0 \times N_0 \to R$, *and* $a(0,t) = b(0,t) = a(s,0) = b(s,0) = 0$. *If* $0 < \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\tau=1}^{m} \sum_{\delta=1}^{n} |\nabla_2 \nabla_1 a(\tau,\delta)|^p < \infty$ *and* $0 < \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \sum_{k=1}^{s} \sum_{r=1}^{t} |\nabla_2 \nabla_1 b(k,r)|^q < \infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m+s)(n+t)} \tag{3.1}$$
$$< \frac{\pi^2}{pq\sin^2(\pi/p)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.$$

In particular, when $p = q = 2$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(mn+st)(m+s)(n+t)}$$
$$< \frac{\pi^2}{2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^2\right\}^{\frac{1}{2}}.$$

Proof. The idea of proof of Theorem 3.1 comes from Theorem 3 in Pachpatte[7] and Theorem 3.1 in Lv[8]. From the hypotheses of Theorem 3.1, it is easy to observe that the following identities hold

$$a(m,n) = \sum_{\tau=1}^{m}\sum_{\delta=1}^{n}\nabla_2\nabla_1 a(\tau,\delta), \quad \text{and} \quad b(s,t) = \sum_{k=1}^{s}\sum_{r=1}^{t}\nabla_2\nabla_1 b(k,r) \tag{3.2}$$

for $(m,n) \in N \times N$, $(s,t) \in N \times N$. From (3.2) and using Hölder's inequality, we have

$$|a(m,n)| \le (mn)^{\frac{1}{q}}\left\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}, \tag{3.3}$$

and

$$|b(s,t)| \le (st)^{\frac{1}{p}}\left\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}} \tag{3.4}$$

for $(m,n) \in N \times N$, $(s,t) \in N \times N$. From (3.3) and (3.4) and using the elementary inequality,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},\ a \ge 0,\ b \ge 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$$

It is easy to observe that

$$
\begin{aligned}
(3.5)\quad & |a(m,n)||b(s,t)| \\
& \le (mn)^{\frac{1}{q}}(st)^{\frac{1}{p}}\left\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}\left\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}} \\
\le & \left(\frac{(mn)^{p-1}}{p}+\frac{(st)^{q-1}}{q}\right)\left\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}\left\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}
\end{aligned}
$$

for $(m,n)\in N\times N$, $(s,t)\in N\times N$. According to (3.5), one obtain

$$
\begin{aligned}
(3.6)\quad \frac{|a(m,n)||b(s,t)|}{q(mn)^{p-1}+p(st)^{q-1}} \le \frac{1}{pq}&\left\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}} \\
&\times\left\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}
\end{aligned}
$$

for $(m,n)\in N\times N$, $(s,t)\in N\times N$. Hence

$$
\begin{aligned}
(3.7)\quad & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m+s)(n+t)} \\
\le & \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\}^{\frac{1}{p}}\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\}^{\frac{1}{q}}}{(m+s)(n+t)}.
\end{aligned}
$$

By Hölder's inequality and Lemma 2.1, we obtain

$$
\begin{aligned}
& \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\}^{\frac{1}{p}}\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\}^{\frac{1}{q}}}{(m+s)(n+t)} \\
= & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\}^{\frac{1}{p}}}{[(m+s)(n+t)]^{\frac{1}{p}}}\left(\frac{mn}{st}\right)^{\frac{1}{pq}} \\
& \times\frac{\{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\}^{\frac{1}{q}}}{[(m+s)(n+t)]^{\frac{1}{q}}}\left(\frac{st}{mn}\right)^{\frac{1}{pq}} \\
\le & \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p}{(m+s)(n+t)}\left(\frac{mn}{st}\right)^{\frac{1}{q}}\right\}^{\frac{1}{p}} \\
& \times\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q}{(m+s)(n+t)}\left(\frac{st}{mn}\right)^{\frac{1}{p}}\right\}^{\frac{1}{q}} \\
< & \frac{\pi^2}{\sin^2(\pi/p)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

By (3.7) and the above inequality, we get (3.1). ∎

In a similar manner to the proof of Theorem 3.1, we can prove the following theorems.

Theorem 5. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $2 - \min\{p, q\} < \lambda_1, \lambda_2 \leq 2 + \min\{p, q\}$. *Let* $a(s, t) : N_0 \times N_0 \to R$, $b(k, r) : N_0 \times N_0 \to R$, *and* $a(0, t) = b(0, t) = a(s, 0) = b(s, 0) = 0$. *If* $0 < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n} m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau, \delta)|^p < \infty$ *and* $0 < \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t} s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k, r)|^q < \infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m+s)^{\lambda_1}(n+t)^{\lambda_2}}$$
$$< \frac{B\left(\frac{p+\lambda_1-2}{p}, \frac{q+\lambda_1-2}{q}\right)B\left(\frac{p+\lambda_2-2}{p}, \frac{q+\lambda_2-2}{q}\right)}{pq}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n} m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t} s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.$$

In particular, when $p = q = 2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(mn+st)(m+s)^{\lambda_1}(n+t)^{\lambda_2}}$$
$$< \frac{B\left(\frac{\lambda_1}{2}, \frac{\lambda_1}{2}\right)B\left(\frac{\lambda_2}{2}, \frac{\lambda_2}{2}\right)}{2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n} m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau,\delta)|^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t} s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k,r)|^2\right\}^{\frac{1}{2}}.$$

Theorem 6. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda_1, \lambda_2 \leq 2$. *Let* $a(s, t) : N_0 \times N_0 \to R$, $b(k, r) : N_0 \times N_0 \to R$, *and* $a(0, t) = b(0, t) = a(s, 0) = b(s, 0) = 0$. *If* $0 < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n} m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau, \delta)|^p < \infty$ *and* $0 < \sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t} s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k, r)|^q < \infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m^{\lambda_1}+s^{\lambda_1})(n^{\lambda_2}+t^{\lambda_2})}$$
$$< \frac{B\left(\frac{p+\lambda_1-2}{p\lambda_1}, \frac{q+\lambda_1-2}{q\lambda_1}\right)B\left(\frac{p+\lambda_2-2}{p\lambda_2}, \frac{q+\lambda_2-2}{q\lambda_2}\right)}{pq\lambda_1\lambda_2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n} m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t} s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.$$

In particular, when $p = q = 2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(mn+st)(m^{\lambda_1}+s^{\lambda_1})(n^{\lambda_2}+t^{\lambda_2})}$$
$$<\frac{\pi^2}{2\lambda_1\lambda_2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau,\delta)|^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k,r)|^2\right\}^{\frac{1}{2}}.$$

Theorem 7. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\lambda_2\leq\min\{p,q\}$. *Let* $a(s,t):N_0\times N_0\to R$, $b(k,r):N_0\times N_0\to R$, *and* $a(0,t)=b(0,t)=a(s,0)=b(s,0)=0$. *If* $0<\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{(p-1)(1-\lambda_1)}n^{(p-1)(1-\lambda_2)}|\nabla_2\nabla_1 a(\tau,\delta)|^p<\infty$ *and* $0<\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{(q-1)(1-\lambda_1)}t^{(q-1)(1-\lambda_2)}|\nabla_2\nabla_1 b(k,r)|^q<\infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\ln\left(\frac{m}{s}\right)\ln\left(\frac{n}{t}\right)|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m^{\lambda_1}-s^{\lambda_1})(n^{\lambda_2}-t^{\lambda_2})}$$
$$<\frac{\pi^4}{pq\lambda_1^2\lambda_2^2\sin^4(\pi/p)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{(p-1)(1-\lambda_1)}n^{(p-1)(1-\lambda_2)}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{(q-1)(1-\lambda_1)}t^{(q-1)(1-\lambda_2)}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$, *we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{\ln\left(\frac{m}{s}\right)\ln\left(\frac{n}{t}\right)|a(m,n)||b(s,t)|}{(mn+st)(m^{\lambda_1}-s^{\lambda_1})(n^{\lambda_2}-t^{\lambda_2})}$$
$$<\frac{\pi^4}{2\lambda_1^2\lambda_2^2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{1-\lambda_1}n^{1-\lambda_2}|\nabla_2\nabla_1 a(\tau,\delta)|^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{1-\lambda_1}t^{1-\lambda_2}|\nabla_2\nabla_1 b(k,r)|^2\right\}^{\frac{1}{2}}.$$

Theorem 8. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\lambda_3<q$, $0<\lambda_2,\lambda_4<q$. *Let* $a(s,t):N_0\times N_0\to R$, $b(k,r):N_0\times N_0\to R$, *and* $a(0,t)=b(0,t)=a(s,0)=b(s,0)=0$. *If* $0<\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{(p-1)(1-\lambda_1)}n^{(p-1)(1-\lambda_3)}|\nabla_2\nabla_1 a(\tau,\delta)|^p<\infty$ *and* $0<\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{(q-1)(1-\lambda_2)}t^{(q-1)(1-\lambda_4)}|\nabla_2\nabla_1 b(k,r)|^q<\infty$. *Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(q(mn)^{p-1}+p(st)^{q-1})(m^{\lambda_1}+s^{\lambda_2})(n^{\lambda_3}+t^{\lambda_4})}$$
$$<\frac{\pi^2}{pq(\lambda_1\lambda_3)^{\frac{1}{q}}(\lambda_2\lambda_4)^{\frac{1}{p}}\sin^2(\pi/p)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{(p-1)(1-\lambda_1)}n^{(p-1)(1-\lambda_3)}|\nabla_2\nabla_1 a(\tau,\delta)|^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{(q-1)(1-\lambda_2)}t^{(q-1)(1-\lambda_4)}|\nabla_2\nabla_1 b(k,r)|^q\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$*, we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\frac{|a(m,n)||b(s,t)|}{(mn+st)(m^{\lambda_1}+s^{\lambda_2})(n^{\lambda_3}+t^{\lambda_4})}$$
$$<\frac{\pi^2}{2\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\tau=1}^{m}\sum_{\delta=1}^{n}m^{1-\lambda_1}n^{1-\lambda_3}|\nabla_2\nabla_1 a(\tau,\delta)|^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}\sum_{k=1}^{s}\sum_{r=1}^{t}s^{1-\lambda_2}t^{1-\lambda_4}|\nabla_2\nabla_1 b(k,r)|^2\right\}^{\frac{1}{2}}.$$

4. Integral analogues

In what follows we let $I=[0,\infty)$ and $I_0=(0,\infty)$ denote the subintervals of R. For the function $u(s,t):I\to R$, we denote the partial derivatives $(\partial/\partial s)u(s,t)$, $(\partial/\partial t)u(s,t)$, $(\partial^2/\partial s\partial t)u(s,t)$ and $(\partial^2/\partial t\partial s)u(s,t)$ by $D_1u(s,t)$, $D_2u(s,t)$, $D_1D_2u(s,t)$ and $D_2D_1u(s,t)$, respectively.

Theorem 9. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$*. Let* $f(\tau,\delta)$ *and* $g(k,r)$ *be real-valued continuous functions defined on* $I\times I$ *and* $I\times I$*, respectively, and* $f(0,\tau)=g(0,\tau)=f(\delta,0)=g(\delta,0)=0$*. If*

$$0<\int_0^{\infty}\int_0^{\infty}\int_0^{x}\int_0^{y}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dx dy<\infty$$

and

$$0<\int_0^{\infty}\int_0^{\infty}\int_0^{s}\int_0^{t}|D_2D_1g(k,r)|^q dk dr ds dt<\infty.$$

Then

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x+s)(y+t)}dsdtdxdy \tag{4.1}$$
$$\leq \frac{\pi^2}{pq\sin^2(\pi/p)}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(xy+st)(x+s)(y+t)}$$
$$\leq \frac{\pi^2}{pq\sin^2(\pi/p)}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^2 d\tau d\delta dxdy\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t |D_2D_1g(k,r)|^2 dkdrdsdt\right\}^{\frac{1}{2}}.$$

Proof. The idea of proof of Theorem 4.1 also comes from Theorem 4 in Pachpatte[7] and Theorem 2.5 in Lv[8]. From the hypotheses of Theorem 4.1, it is easy to observe that the following identities hold

$$f(x,y)=\int_0^x\int_0^y D_2D_1f(\tau,\delta)d\tau d\delta,\quad\text{and}\quad g(s,t)=\int_0^s\int_0^t D_2D_1g(k,r)dkdr \tag{4.2}$$

for $(x,y)\in I\times I$, $(s,t)\in I\times I$. From (4.2) and using Hölder's inequality, we have

$$|f(x,y)|\leq (xy)^{\frac{1}{q}}\left\{\int_0^x\int_0^y |D_2D_1a(\tau,\delta)|^p d\tau d\delta\right\}^{\frac{1}{p}}, \tag{4.3}$$

and

$$|g(s,t)|\leq (st)^{\frac{1}{p}}\left\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\right\}^{\frac{1}{q}} \tag{4.4}$$

for $(x,y) \in I \times I$, $(s,t) \in I \times I$. From (4.3)and (4.4) and using the elementary inequality, it is easy to observe that

$$
\begin{aligned}
(4.5)\quad |f(x,y)||g(s,t)| &\le (xy)^{\frac{1}{q}}(st)^{\frac{1}{p}}\left\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\right\}^{\frac{1}{q}} \\
&\le \left(\frac{(xy)^{p-1}}{p}+\frac{(st)^{q-1}}{q}\right)\left\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\right\}^{\frac{1}{q}}
\end{aligned}
$$

for $(x,y) \in I \times I$, $(s,t) \in I \times I$. According to (4.5), one obtain

$$
\begin{aligned}
(4.6)\quad \frac{|f(x,y)||g(s,t)|}{q(xy)^{p-1}+p(st)^{q-1}} &\le \frac{1}{pq}\left\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\right\}^{\frac{1}{p}} \\
&\quad\times\left\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\right\}^{\frac{1}{q}}
\end{aligned}
$$

for $(x,y) \in I \times I$, $(s,t) \in I \times I$. Hence

$$
\begin{aligned}
(4.7)\quad &\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x+s)(y+t)}dsdtdxdy \\
&\le \frac{1}{pq}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\}^{\frac{1}{p}}\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\}^{\frac{1}{q}}}{(x+s)(y+t)}dsdtdxdy.
\end{aligned}
$$

By Hölder's inequality and Lemma 2.1, we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\}^{\frac{1}{p}}\{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr\}^{\frac{1}{q}}}{(x+s)(y+t)} dsdtdxdy$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta\}^{\frac{1}{p}}}{[(x+s)(y+t)]^{\frac{1}{p}}}\left(\frac{xy}{st}\right)^{\frac{1}{pq}}$$

$$\times \frac{\{\int_0^s\int_0^t |D_2D_1b(k,r)|^q dkdr\}^{\frac{1}{q}}}{[(x+s)(y+t)]^{\frac{1}{q}}}\left(\frac{st}{xy}\right)^{\frac{1}{pq}} dsdtdxdy$$

$$\leq \left\{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta}{(x+s)(y+t)}\left(\frac{xy}{st}\right)^{\frac{1}{q}} dsdtdxdy\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdr}{(x+s)(y+t)}\left(\frac{st}{xy}\right)^{\frac{1}{p}} dsdtdxdy\right\}^{\frac{1}{q}}$$

$$\leq \frac{\pi^2}{\sin^2(\pi/p)}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y |D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t |D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

By (4.7) and the above inequality, we get (4.1). This completes the proof of Theorem 4.1. ∎

In a similar manner to the proof of Theorem 4.1, we can prove the following theorems.

Theorem 10. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $2 - \min\{p, q\} < \lambda_1, \lambda_2$. *Let* $f(\tau, \delta)$ *and* $g(k, r)$ *be real-valued continuous functions defined on* $I \times I$ *and* $I \times I$, *respectively, and* $f(0, \tau) = g(0, \tau) = f(\delta, 0) = g(\delta, 0) = 0$. *If*

$$0 < \int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy < \infty$$

and

$$0 < \int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^q dkdrdsdt < \infty.$$

Then

$$(4.8)\quad \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x+s)^{\lambda_1}(y+t)^{\lambda_2}}dsdtdxdy$$
$$\leq \frac{B\left(\frac{p+\lambda_1-2}{p},\frac{q+\lambda_1-2}{q}\right)B\left(\frac{p+\lambda_2-2}{p},\frac{q+\lambda_2-2}{q}\right)}{pq}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$*, we have*

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(xy+st)(x+s)^{\lambda_1}(y+t)^{\lambda_2}}dsdtdxdy$$
$$\leq \frac{B\left(\frac{\lambda_1}{2},\frac{\lambda_1}{2}\right)B\left(\frac{\lambda_2}{2},\frac{\lambda_2}{2}\right)}{2}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^2 d\tau d\delta dxdy\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^2 dkdrdsdt\right\}^{\frac{1}{2}}.$$

Theorem 11. *Let* $p>1$*,* $\frac{1}{p}+\frac{1}{q}=1$*,* $2-\min\{p,q\}<\lambda_1,\lambda_2$*. Let* $f(\tau,\delta)$ *and* $g(k,r)$ *be real-valued continuous functions defined on* $I\times I$ *and* $I\times I$*, respectively, and* $f(0,\tau)=g(0,\tau)=f(\delta,0)=g(\delta,0)=0$*. If*

$$0<\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy<\infty$$

and

$$0<\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^q dkdrdsdt<\infty.$$

Then

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x^{\lambda_1}+s^{\lambda_1})(y^{\lambda_2}+t^{\lambda_2})}dsdtdxdy$$
$$\leq \frac{B\left(\frac{p+\lambda_1-2}{p\lambda_1},\frac{q+\lambda_1-2}{q\lambda_1}\right)B\left(\frac{p+\lambda_2-2}{p\lambda_2},\frac{q+\lambda_2-2}{q\lambda_2}\right)}{pq\lambda_1\lambda_2}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(xy+st)(x^{\lambda_1}+s^{\lambda_1})(y^{\lambda_2}+t^{\lambda_2})}dsdtdxdy$$
$$\leq \frac{\pi^2}{2\lambda_1\lambda_2}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^2 d\tau d\delta dxdy\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^2 dkdrdsdt\right\}^{\frac{1}{2}}.$$

Theorem 12. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\lambda_2$. Let $f(\tau,\delta)$ and $g(k,r)$ be real-valued continuous functions defined on $I\times I$ and $I\times I$, respectively, and $f(0,\tau)=g(0,\tau)=f(\delta,0)=g(\delta,0)=0$. If*

$$0<\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{(p-1)(1-\lambda_1)}y^{(p-1)(1-\lambda_2)}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy<\infty$$

and

$$0<\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{(q-1)(1-\lambda_1)}t^{(q-1)(1-\lambda_2)}|D_2D_1g(k,r)|^q dkdrdsdt<\infty.$$

Then

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\ln\left(\frac{x}{s}\right)\ln\left(\frac{y}{t}\right)|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x^{\lambda_1}-s^{\lambda_1})(y^{\lambda_2}-t^{\lambda_2})}$$

$$\leq \frac{\pi^4}{pq\lambda_1^2\lambda_2^2\sin^4(\pi/p)}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{(p-1)(1-\lambda_1)}y^{(p-1)(1-\lambda_2)}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{(q-1)(1-\lambda_1)}t^{(q-1)(1-\lambda_2)}|D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{\ln\left(\frac{x}{s}\right)\ln\left(\frac{y}{t}\right)|f(x,y)||g(s,t)|}{(xy+st)(x^{\lambda_1}-s^{\lambda_1})(y^{\lambda_2}-t^{\lambda_2})}$$

$$\leq \frac{\pi^4}{2\lambda_1^2\lambda_2^2}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_2}|D_2D_1f(\tau,\delta)|^2 d\tau d\delta dxdy\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_1}t^{1-\lambda_2}|D_2D_1g(k,r)|^2 dkdrdsdt\right\}^{\frac{1}{2}}.$$

Theorem 13. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda_1,\lambda_2,\lambda_3,\lambda_4$. Let $f(\tau,\delta)$ and $g(k,r)$ be real-valued continuous functions defined on $I\times I$ and $I\times I$, respectively, and $f(0,\tau)=g(0,\tau)=f(\delta,0)=g(\delta,0)=0$. If*

$$0<\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{(p-1)(1-\lambda_1)}y^{(p-1)(1-\lambda_3)}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy<\infty$$

and

$$0<\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{(q-1)(1-\lambda_2)}t^{(q-1)(1-\lambda_4)}|D_2D_1g(k,r)|^q dkdrdsdt<\infty.$$

Then

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(q(xy)^{p-1}+p(st)^{q-1})(x^{\lambda_1}+s^{\lambda_2})(y^{\lambda_3}+t^{\lambda_4})}\leq \frac{\pi^2}{pq(\lambda_1\lambda_3)^{\frac{1}{q}}(\lambda_2\lambda_4)^{\frac{1}{p}}\sin^2(\pi/p)}$$

$$\times\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{(p-1)(1-\lambda_1)}y^{(p-1)(1-\lambda_3)}|D_2D_1f(\tau,\delta)|^p d\tau d\delta dxdy\right\}^{\frac{1}{p}}$$

$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{(q-1)(1-\lambda_2)}t^{(q-1)(1-\lambda_4)}|D_2D_1g(k,r)|^q dkdrdsdt\right\}^{\frac{1}{q}}.$$

In particular, when $p=q=2$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{|f(x,y)||g(s,t)|}{(xy+st)(x^{\lambda_1}+s^{\lambda_2})(y^{\lambda_3}+t^{\lambda_4})}$$
$$\leq \frac{\pi^2}{2\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}}\left\{\int_0^\infty\int_0^\infty\int_0^x\int_0^y x^{1-\lambda_1}y^{1-\lambda_3}|D_2D_1f(\tau,\delta)|^2 d\tau d\delta dx dy\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_0^\infty\int_0^\infty\int_0^s\int_0^t s^{1-\lambda_2}t^{1-\lambda_4}|D_2D_1g(k,r)|^2 dk dr ds dt\right\}^{\frac{1}{2}}.$$

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