ANOTHER PROOF OF LEVINSON INEQUALITY

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ABSTRACT. New proofs of Levinson inequality are presented.

In 1964 Norman Levinson ([2]) used Taylor expansion to prove the following

Theorem 1.1. *Suppose that* $f : [0, 2b] \to \mathbf{R}$ *has a nonnegative third derivative, for* $i = 1, \dots, n$ $p_i > 0$, $0 \le x_i \le b$, $y_i = 2b - x_i$ *and* $\sum_{i=1}^n p_i = 1$, *then the inequality*

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(y_i) - f\left(\sum_{i=1}^n p_i y_i\right) \tag{1.1}$$

holds.

The same year Tiberiu Popiviciu generalised it by showing that for (1.1) to hold it is enough if f is 3-convex ([3]) and Peter S. Bullen simplified his proof using mathematical induction. This is the final version:

Theorem 1.2. *If* $f : [a, b] \to \mathbf{R}$ *is* 3*-convex,* $a \le x_i, y_i \le b$, $x_i + y_i = c$, $p_i > 0$ *for* $i = 1, \dots, n$, $\sum_{i=1}^n p_i = 1$ *and*

$$\max(x_1, \dots, x_n) \le \min(y_1, \dots, y_n), \tag{1.2}$$

then (1.1) *holds.*

The aim of this note is to give two simple proofs of theorem 1.2 based on the fact that any 3-convex function is differentiable and its first derivative is convex ([1]).

First proof. Denote $\overline{x} = \sum_{i=1}^n p_i x_i$ and $\overline{y} = \sum_{i=1}^n p_i y_i = c - \overline{x}$. For $0 \le t \le 1$ let $x_i(t) = \overline{x} + t(x_i - \overline{x})$ and $y_i(t) = \overline{y} + t(y_i - \overline{y}) = c - x_i(t)$. The function

$$U(t) = \sum_{i=1}^n p_i f(y_i(t)) - f(\overline{y}) + \sum_{i=1}^n p_i f(x_i(t)) - f(\overline{x})$$

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is differentiable and $U(0) = 0$. Since $U(1) \geq 0$ is equivalent to (1.1), all we have to do is to show that U is nondecreasing.

$$\begin{aligned}
U'(t) &= \sum_{i=1}^{n} p_i(y_i - \overline{y}) f'(y_i(t)) - \sum_{i=1}^{n} p_i(x_i - \overline{x}) f'(x_i(t)) \\
&= \sum_{i=1}^{n} p_i(y_i - \overline{y})(f'(y_i(t)) - f'(\overline{y})) - \sum_{i=1}^{n} p_i(x_i - \overline{x})(f'(x_i(t)) - f'(\overline{x})) \\
&= \sum_{i=1}^{n} p_i(y_i - \overline{y})^2 \frac{f'(y_i(t)) - f'(\overline{y})}{y_i - \overline{y}} - \sum_{i=1}^{n} p_i(x_i - \overline{x})^2 \frac{f'(x_i(t)) - f'(\overline{x})}{x_i - \overline{x}} \\
&= t \sum_{i=1}^{n} p_i(y_i - \overline{y})^2 \left[\frac{f'(y_i(t)) - f'(\overline{y})}{y_i(t) - \overline{y}} - \frac{f'(x_i(t)) - f'(\overline{x})}{x_i(t) - \overline{x}} \right] \geq 0
\end{aligned}$$

The last inequality is valid because $|x_i - \overline{x}| = |y_i - \overline{y}|$, divided difference of f' is inreasing due to its convexity and (1.2) implies, that all y's are greater than x's.

□

Second proof. Writing (1.1) in the form

$$f\left(c - \sum_{i=1}^{n} p_i x_i\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i \left[f(c - x_i) - f(x_i)\right]$$

we see that this is nothing but Jensen's inequality applied to the function $g(x) = f(c - x) - f(x)$, so our goal would be to show, that g is convex for $x < c/2$ (the last condition follows from (1.2)). We shall achieve this by showing that its derivative is nondecreasing. For arbitrary $x, y < c/2$

$$\begin{aligned}
\frac{g'(y) - g'(x)}{y - x} &= \frac{-f'(c - y) - f'(y) + f'(c - x) + f'(x)}{y - x} \\
&= \frac{f'(c - x) - f'(c - y)}{(c - x) - (c - y)} - \frac{f'(y) - f'(x)}{y - x} \geq 0
\end{aligned}$$

since $c - x > x$ and $c - y > y$. □

References

[1] R.P. Boas and D.V. Widder, *Functions with positive differences*, Duke Math. J. **7** (1940) 496-503
[2] N. Levinson, *Generalization of an inequality of Ky Fan*, J. Math. Anal. Appl. **8** (1964), 133-134
[3] T. Popoviciu, *Sur une inegalité de N. Levinson*. Mathematica (Cluj) **6** (1964), 301-306.

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