New inequalities similar to Ostrowski's inequality *

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May 6, 2009

Abstract

In the present note we establish new inequalities similar to that of the well known Ostrowski's inequality by using a fairly elementary analysis.

1 Introduction

One of the most remarkable results, proved by A.M.Ostrowski [5] is the following inequality (see also [4,p.468]):

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \qquad (1.1)$$

for all $x \in [a,b]$, where $f : [a,b] \subseteq R \to R$ be continuous on $[a,b]$ and differentiable on (a,b) , whose derivative $f' : (a,b) \to R$ is bounded on (a,b) , i.e., $\|f'\|_\infty = \sup_{t \in (a,b)} |f'(t)| < \infty$. During the last few years, a variety of Ostrowski type inequalities have been investigated and used in the approximation theory and numerical analysis, see [2,4] and the references cited therein. In 1995 G.A.Anastassiou [1] proved the following Ostrowski type inequality:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \le \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+2)!} \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right), \qquad (1.2)$$

where $f \in C^{n+1}([a,b]), n \in N$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0, k = 1, ..., n$.

The main objective of the present note is to establish new inequalities of the type given in (1.2) involving two functions and their higher order derivatives. The analysis used in the proofs is based on the Taylor's formula with the integral reminder.

* *2000 Mathematics Subject Classification:* 26D10,26D15

Key words and Phrases: Ostrowski's inequality, differentiable, higher order derivatives, Taylor's formula.

2 Statement of Results

In what follows, we denote by R the set of real numbers and let $[a,b] \subseteq R$ and N be the set of natural numbers. The proofs of the results given below are based on the following version of the Taylor's formula given in [1,p.3777] (see also [3]):

$$h(y) - h(x) = \sum_{k=1}^{n} \frac{h^{(k)}(x)}{k!} (y-x)^k + R_n(x,y), \tag{2.1}$$

where $h \in C^{n+1}([a,b], R), n \in N$ and $x \in [a,b]$ be fixed and

$$R_n(x,y) = \int_x^y \left(h^{(n)}(t) - h^{(n)}(x)\right) \frac{(y-t)^{n-1}}{(n-1)!} dt, \tag{2.2}$$

here $y \geq x$ or $y \leq x$.

Our main result is given in the following theorem.

Theorem 1 . Let $f, g \in C^{n+1}([a,b], R), n \in N$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0, g^{(k)}(x) = 0, k = 1, ..., n$. Then

$$\left| 2\int_a^b f(y)\, g(y)\, dy - \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] \right|$$

$$\leq \frac{1}{(n+1)!} \int_a^b \left[|g(y)| \left\| f^{(n+1)} \right\|_\infty + |f(y)| \left\| g^{(n+1)} \right\|_\infty \right] |y-x|^{n+1} dy. \tag{2.3}$$

A slightly different version of Theorem 1 is embodied in the following theorem.

Theorem 2 . Let f, g, n, x be as in Theorem 1. Then

$$\left| \int_a^b f(y)\, g(y)\, dy - \left[g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right] + (b-a) f(x) g(x) \right|$$

$$\leq \left\{ \frac{1}{(n+1)!} \right\}^2 \left\| f^{(n+1)} \right\|_\infty \left\| g^{(n+1)} \right\|_\infty \left(\frac{(x-a)^{2n+3} + (b-x)^{2n+3}}{2n+3} \right). \tag{3.4}$$

3 Proofs of Theorems 1 and 2

From the hypotheses of Theorems 1 and 2 and using Taylor's formula (2.1) we have

$$f(y) - f(x) = A_n(x, y), \tag{3.1}$$

$$g(y) - g(x) = B_n(x, y), \tag{3.2}$$

where

$$A_n(x, y) = \int_x^y \left(f^{(n)}(t) - f^{(n)}(x)\right) \frac{(y-t)^{n-1}}{(n-1)!} dt, \tag{3.3}$$

$$B_n(x, y) = \int_x^y \left(g^{(n)}(t) - g^{(n)}(x)\right) \frac{(y-t)^{n-1}}{(n-1)!} dt, \tag{3.4}$$

here $y \geq x$ or $y \leq x$. Multiplying both sides of (3.1) and (3.2) by $g(y)$ and $f(y)$ respectively and adding we get

$$2f(y) g(y) - [g(x) f(y) + f(x) g(y)] = g(y) A_n(x, y) + f(y) B_n(x, y). \tag{3.5}$$

Integrating both sides of (3.5) with respect to y over $[a, b]$ we have

$$2\int_a^b f(y) g(y) dy - \left[g(x)\int_a^b f(y) dy + f(x)\int_a^b g(y) dy\right]$$

$$= \int_a^b [g(y) A_n(x, y) + f(y) B_n(x, y)] dy. \tag{3.6}$$

From (3.6) and using the properties of modulus we have

$$\left|2\int_a^b f(y) g(y) dy - \left[g(x)\int_a^b f(y) dy + f(x)\int_a^b g(y) dy\right]\right|$$

$$\leq \int_a^b [|g(y)| |A_n(x, y)| + |f(y)| |B_n(x, y)|] dy. \tag{3.7}$$

By following the same arguments as given in [1,p.3777] we have

$$|A_n(x, y)| \leq \frac{\left\|f^{(n+1)}\right\|_\infty}{(n+1)!} |y - x|^{n+1}, \tag{3.8}$$

$$|B_n(x, y)| \leq \frac{\left\|g^{(n+1)}\right\|_\infty}{(n+1)!} |y - x|^{n+1}. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7) we get the desired inequality in (2.3). The proof of Theorem 1 is complete.

Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$f(y)\,g(y) - [g(x)\,f(y) + f(x)\,g(y)] + f(x)\,g(x)$$
$$= A_n(x,y)\,B_n(x,y)\,. \tag{3.10}$$

Integrating both sides of (3.10) with respext to y over $[a,b]$ we have

$$\int_a^b f(y)\,g(y)\,dy - \left[g(x)\int_a^b f(y)\,dy + f(x)\int_a^b g(y)\,dy\right] + (b-a)\,f(x)\,g(x)$$
$$= \int_a^b A_n(x,y)\,B_n(x,y)\,dy. \tag{3.11}$$

From (3.11) and using the properties of modulus, (3.8),(3.9) and simple calculation we get the required inequality in (2.4). The proof of Theorem 2 is complete.

We note that in the special case, if we take $g(t) = 1$ and hence $g^{(n+1)}(t) = 0$ in Theorem 1, then by simple calculation we get the inequality given by Anastassiou in [1].

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