On some discrete inequalities of Grüss type *

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Abstract

In the present paper we establish some new Grüss type discrete inequalities by using a fairly elementary analysis .

1 Introduction

In 1935 , G. Grüss [4] proved the following integral inequality (see also [5 , p.296])

$$\left| \frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right) \right|$$
$$\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \tag{1.1}$$

where $f, g : [a,b] \to R$ are integrable on $[a,b]$ and satisfying the assumption

$$\phi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma,$$

for each $x \in [a,b]$ where $\phi, \Phi, \gamma, \Gamma$ are given real constants .

In 1950 , M.Biernacki , H.Pidek and C.Ryll-Nardzewski established the following discrete version of Grüss inequality [5 , Chapter X].

Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ be two n-tuple of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, ..., n$. Then

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right) \left(\frac{1}{n} \sum_{i=1}^{n} b_i \right) \right|$$
$$\leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s), \tag{1.2}$$

where $[x]$ is the integer part of x , $x \in R$.

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In 1979 , J.E.Pečarić established the following weighted version of Grüss discrete inequality [5 , Chapter X].

Let a, b and p be three monotonic n-tuples with all elements of p positive . Then

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right) \right|$$
$$\leq |a_n - a_1| \, |b_n - b_1| \max_{1 \leq k \leq n-1} \left(\frac{P_k \bar{P}_{k+1}}{P_n^2} \right), \tag{1.3}$$

where $P_n = \sum\limits_{i=1}^{n} p_i, \bar{P}_{k+1} = P_n - P_{k+1}$.

A large number of extensions , generalizations and the discrete variants of the inequality (1.1) have appeared in the literature , see the book [5] by Mitrinović , Pečarić and Fink , where further references are also given .New results in the domain can be found in papers [1-3 , 6-8] . The main aim of the present paper is to establish some new Grüss type discrete inequalities similar to the inequalities given in (1.2) and (1.3) . The analysis used in the proofs is elementary and our results provide new estimates on these types of inequalities .

2 Statement of Results

Our main results are established in the following theorems.

Theorem 1 . Let a_i, b_i $(i = 1, ..., n)$ be real numbers and p_i $(i = 1, ..., n)$ be nonnegative real numbers such that $P_n = \sum\limits_{i=1}^{n} p_i > 0$. Then

$$|T(P_n, p_i, a_i, b_i; 1, n)| \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \left| \left(a_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) \left(b_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j b_j \right) \right|, \tag{2.1}$$

where

$$T(P_n, p_i, a_i, b_i; 1, n) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right). \tag{2.2}$$

Remark 1 . By taking $p_i = 1$ for $i = 1, ..., n$ and hence $P_n = 1$ in (2.1) we get

$$|T(n, 1, a_i, b_i; 1, n)| \leq \frac{1}{n} \sum_{i=1}^{n} \left| \left(a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right) \left(b_i - \frac{1}{n} \sum_{j=1}^{n} b_j \right) \right|. \tag{2.3}$$

We note that the inequality (2.3) can be considered as the discrete analogue of the integral inequality of Grüss type given by Dragomir and McAndrew in [3].

Theorem 2 . Let $a, A, a_i, b, B, b_i\,(i = 1, ..., n)$ be real numbers and $p_i\,(i = 1, ..., n)$ are nonnegative real numbers and $P_n = \sum\limits_{i=1}^{n} p_i > 0,\ a \leq a_i \leq A,\ b \leq b_i \leq B$. Then

$$|T\,(P_n, p_i, a_i, b_i; 1, n)| \leq \frac{1}{4}\,(A - a)\,(B - b)\,, \tag{2.4}$$

where $T\,(P_n, p_i, a_i, b_i; 1, n)$ is given by (2.2).

Remark 2 . If we take $p_i = 1\,(i = 1, ..., n)$ and hence $P_n = n$ in (2.4) , then we get

$$|T\,(n, 1, a_i, b_i; 1, n)| \leq \frac{1}{4}\,(A - a)\,(B - b)\,. \tag{2.5}$$

As noted by Dragomir in [2,p.1043] , the inequality (2.5) can be considered as a discrete analogue of the Grüss inequality (1.1).

Theorem 3 . Let $a_i, b_i\,, p_i\,(i = 1, ..., n)$ and P_n be as in Theorem 2 . Suppose that $|\Delta a_i| \leq L$, $|\Delta b_i| \leq M$, where L, M are nonnegative constants and $\Delta a_i = a_{i+1} - a_i, \Delta b_i = b_{i+1} - b_i$. Then

$$|T\,(P_n, p_i, a_i, b_i; 1, n)| \leq LM \left[\frac{1}{P_n} \sum_{i=1}^{n} p_i i^2 - \left(\frac{\sum\limits_{i=1}^{n} i p_i}{P_n} \right)^2 \right], \tag{2.6}$$

where $T\,(P_n, p_i, a_i, b_i; 1, n)$ is given by (2.2).

Remark 3 . We note that A.Lupas [5 , Chapter X] proved some results similar to that of the inequality (2.6) when $a = (a_1, ..., a_n)\,, b = (b_1, ..., b_n)$ are two monotonic n-tuples in the same sense and $p = (p_1, ..., p_n)$ is a positive n-tuple. In the special case , when $p_i = 1\ (i = 1, ..., n)$ and hence $p_n = n$, the inequality (2.6) reduces to

$$|T\,(n, 1, a_i, b_i; 1, n)| \leq LM \left[\frac{1}{n} \sum_{i=1}^{n} i^2 - \left(\frac{\sum\limits_{i=1}^{n} i}{n} \right)^2 \right]. \tag{2.7}$$

3 Proofs of Theorems 1-3

In order to establish the inequality (2.1) , we first observe that

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \left(a_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j a_j \right) \left(b_i - \frac{1}{P_n} \sum_{j=1}^{n} p_j b_j \right)$$

$$= \frac{1}{P_n}\sum_{i=1}^{n} p_i \left\{ a_i b_i - a_i \frac{1}{P_n}\sum_{j=1}^{n} p_j b_j - b_i \frac{1}{P_n}\sum_{j=1}^{n} p_j a_j \right.$$

$$\left. + \frac{1}{P_n^2}\sum_{j=1}^{n} p_j a_j \sum_{j=1}^{n} p_j b_j \right\}$$

$$= \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j b_j\right)$$

$$- \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j a_j\right)$$

$$+ \frac{1}{P_n}\sum_{i=1}^{n} p_i \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j a_j\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j b_j\right)$$

$$= \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i\right)\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right)$$

$$= T\left(P_n, p_i, a_i, b_i; 1, n\right). \tag{3.1}$$

From (3.1) and using the properties of modulus we have

$$\left|T\left(P_n, p_i, a_i, b_i; 1, n\right)\right| \le \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|\left(a_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j a_j\right)\left(b_i - \frac{1}{P_n}\sum_{j=1}^{n} p_j b_j\right)\right|.$$

This completes the proof of the inequality (2.1).

From the hypotheses of Theorem 2 , by direct computation , it is easy to observe that the following discrete identity holds

$$T\left(P_n, p_i, a_i, b_i; 1, n\right) = \frac{1}{2P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left(a_i - a_j\right)\left(b_i - b_j\right). \tag{3.2}$$

By taking square on the right hand side of (3.2) and applying Schwarz's inequality for double sums we have

$$\left[\frac{1}{2P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left(a_i - a_j\right)\left(b_i - b_j\right)\right]^2$$

$$\le \left[\frac{1}{2P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left(a_i - a_j\right)^2\right]\left[\frac{1}{2P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left(b_i - b_j\right)^2\right]$$

$$= \left[\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i^2 - \left(\frac{\sum_{i=1}^{n} p_i a_i}{P_n} \right)^2 \right] \left[\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i^2 - \left(\frac{\sum_{i=1}^{n} p_i b_i}{P_n} \right)^2 \right]. \tag{3.3}$$

It is easy to observe that the following identity also holds

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i^2 - \left(\frac{\sum_{i=1}^{n} p_i a_i}{P_n} \right)^2 = \left(A - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i - a \right)$$

$$- \frac{1}{P_n} \sum_{i=1}^{n} p_i \left(A - a_i \right) \left(a_i - a \right). \tag{3.4}$$

Using the fact that $\left(A - a_i \right) \left(a_i - a \right) \geq 0$ in (3.4) we have

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i^2 - \left(\frac{\sum_{i=1}^{n} p_i a_i}{P_n} \right)^2 \leq \left(A - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i - a \right). \tag{3.5}$$

Similarly , we have

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i^2 - \left(\frac{\sum_{i=1}^{n} p_i b_i}{P_n} \right)^2 \leq \left(B - \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i - b \right). \tag{3.6}$$

From (3.2) , (3,3) , (3.5) , (3.6) we get

$$\left| T \left(P_n, p_i, a_i, b_i; 1, n \right) \right|^2 \leq \left(A - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i - a \right)$$

$$\times \left(B - \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i b_i - b \right). \tag{3.7}$$

By using the elementary inequality

$$cd \leq \left(\frac{c+d}{2} \right)^2, c, d \in R,$$

we observe that

$$\left(A - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \right) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i a_i - a \right) \leq \left(\frac{A - a}{2} \right)^2, \tag{3.8}$$

and

$$\left(B-\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\right)\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i-b\right)\leq\left(\frac{B-b}{2}\right)^2. \tag{3.9}$$

The desired inequality in (2.4) follows from (3.7) , (3.8) and (3.9) . The proof of Theorem 2 is complete.

From the hypotheses of Theorem 3 , the identity (3.2) holds . It is easy to observe that the following identities also hold

$$a_i-a_j=\sum_{k=j}^{i-1}(a_{k+1}-a_k)=\sum_{k=j}^{i-1}\Delta a_k, \tag{3.10}$$

$$b_i-b_j=\sum_{k=j}^{i-1}(b_{k+1}-b_k)=\sum_{k=j}^{i-1}\Delta b_k, \tag{3.11}$$

for $i>j$. Using (3.10) and (3.11) in (3.2) we have

$$T(P_n,p_i,a_i,b_i;1,n)=\frac{1}{2P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j\left(\sum_{k=j}^{i-1}\Delta a_k\right)\left(\sum_{k=j}^{i-1}\Delta b_k\right). \tag{3.12}$$

From (3.12) and using the properties of modulus and the hypotheses we have

$$|T(P_n,p_i,a_i,b_i;1,n)|\leq\frac{1}{2P_n^2}LM\sum_{i=1}^{n}\sum_{j=1}^{n}p_ip_j|i-j|^2$$

$$=\frac{1}{2P_n^2}LM2\left[P_n\sum_{i=1}^{n}p_ii^2-\left(\sum_{i=1}^{n}ip_i\right)^2\right]$$

$$=LM\left[\frac{1}{P_n}\sum_{i=1}^{n}p_ii^2-\left(\frac{\sum_{i=1}^{n}ip_i}{P_n}\right)^2\right].$$

This completes the proof of Theorem 3 .

In concluding , we note that in [6] the author gave multidimensional Gtüss type inequalities whose proofs were based on a certain finite difference identity . Here we note that the inequalities established in Theorems 1-3 are different and can not be obtained as special cases of those given in [6] . For some other Grüss type discrete inequalities , see [2 , 7 , 8] and the references given therein .

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