ABOUT BERGSTROM AND DUNKL-WILLIAMS INEQUALITIES IN PSEUDO-HILBERT SPACES

LOREDANA CIURDARIU

Abstract. A variant of Bergstrom inequality for gramian normal operators in pseudo-Hilbert spaces and Hilbert spaces and some consequences are presented.

1. Introduction

Let Z be an admissible space in the Loynes sense and $\mathcal{H}$ be a Loynes Z-space (pseudo-Hilbert space), see [3, 4, 5].

It is known, see [3], that if p is a continuous and monotonous seminorm on Z, then $q_p(h) = (p([h,h]))^{1/2}$ is a continuous seminorm on $\mathcal{H}$.

Also, see [3], if $\mathcal{H}$ is a pre-Loynes Z-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of Z, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathcal{P}} = \{q_p \mid p \in \mathcal{P}\}$.

Let $\mathcal{H}$ be a Loynes Z- space, where Z is an admissible space. We shall recall two results which will be used below in this paper, in the proof of Theorem 2.

Proposition 1. [3] *For an operator $N \in \mathcal{C}^*(\mathcal{H})$ the following assertions are equivalent:*

(i): *N is gramian normal;*
(ii): *$[Nh, Nh] = [N^*h, N^*h]$, $h \in \mathcal{H}$;*
(iii): *There exist two gramian self-adjoint commutative operators $A, B \in \mathcal{L}_h^*(\mathcal{H})$, such that $N = A + iB$.*

Theorem 1. [3] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Loynes Z–spaces and $N_1, N_2 \in \mathcal{B}^*(\mathcal{H})$ two gramian normal operators on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. If there exists $T \in \mathcal{B}^*(\mathcal{H}_1, \mathcal{H}_2)$ which inverts N_1 and N_2 , that is $TN_1 = N_2T$, then T also inverts the adjoints N_1^* and N_2^*, i.e. $TN_1^* = N_2^*T$.*

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2. The main results

We shall consider again $\mathcal{H}$ as being a Loynes Z space.

Remark 1. *For every operators $A, B \in \mathcal{B}^*(\mathcal{H})$ and $a,\ b > 0$, we have*

$$\frac{A^2}{a} + \frac{B^2}{b} \geq \frac{(A+B)^2}{a+b}.$$

Proof. The obviously inequality

$$(\sqrt{\frac{b}{a}}A - \sqrt{\frac{a}{b}}B)^2 \geq 0$$

leads to

$$A^2 + B^2 + \frac{b}{a}A^2 + \frac{a}{b}B^2 \geq A^2 + B^2 + AB + BA$$

or

$$(1 + \frac{b}{a})A^2 + (1 + \frac{a}{b})B^2 \geq (A+B)^2$$

i.e.

$$\frac{(a+b)}{a}A^2 + \frac{(a+b)}{b}B^2 \geq (A+B)^2.$$

□

Lemma 1. *If $n \in N,\ n \leq 2,\ N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ are n gramian normal commutative operators then,*

$$|\sum_{k=1}^{n} N_k|^2 = \sum_{k=1}^{n} |N_k|^2 + \frac{1}{4}\sum_{i,j=1}^{n}[(N_i + N_i^*)(N_j + N_j^*) - (N_i - N_i^*)(N_j - N_j^*)],$$

where $|N| = (N^*N)^{\frac{1}{2}}$.

We start this section by given a generalization of the identity (4) from Theorem 3, see [9] for a particular class of operators on Loynes spaces. In the proof we will use the same techniques as in [9]. There is also a proof of this theorem by using induction.

Theorem 2. *If $n \in N,\ n \leq 2,\ N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ are n gramian normal commutative operators and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n}\sum_{1 \leq i < j \leq n}\frac{|a_iN_j - a_jN_i|^2}{a_ia_j}, \tag{1}$$

where $|N| = (N^*N)^{\frac{1}{2}}$, it is known as modulus of N.

Proof. If the operators $N_i,\ i \in \{1, ..., n\}$ are gramian normal then from [3], Proposition 1.2.2, we have the following decompositions for $N_k,\ k \in \{1, ..., n\},\ N_k = A_k + iB_k$ with $A_k,\ B_k$ being gramian commutative self-adjoint operators.

We mention that if the operators $N_k,\ k \in \{1, ..., n\}$ are gramian normal and gramian commutative as pairs then also will be and the operators $A_i,\ B_j$, that means $A_iA_k = A_kA_i,\ A_iB_k = B_kA_i,\ B_kB_i = B_iB_k,\ k, i \in \{1, ..., n\}$, by using the particular form of operators $A_i,\ B_i$ in the decomposition of N_i and the Fuglede's theorem, Theorem 1, for $N_i,\ N_k,\ k \in \{1,...,n\}$.

In fact from the proof of Proposition 1, $A_k = \frac{1}{2}(N_k + N_k^*)$ and $B_k = \frac{1}{2i}(N_k - N_k^*)$, and by the Theorem 1 applied for the gramian commutative normal operators N_i we have from $N_k N_k^* = N_k^* N_k,\ N_i N_k = N_k N_i,\ N_i N_k^* = N_k^* N_i$ and $N_i^* N_k^* = N_k^* N_i^*,\ (\forall) i,k \in \{1,...,n\}$ the above equalities.

Thus $|N_k|^2 = |A_k + iB_k|^2 = (A_k + iB_k)^*(A_k + iB_k) = (A_k - iB_k)(A_k + iB_k) = A_k^2 + B_k^2,\ k \in \{1,...,n\}$,

$$|N_1 + N_2 + ... + N_n|^2 = |(A_1 + A_2 + ... + A_n) + i(B_1 + B_2 + ... + B_n)|^2 =$$
$$= (A_1 + A_2 + ... + A_n)^2 + (B_1 + B_2 + ... + B_n)^2$$

Moreover,

$$|a_k N_j - a_j N_k|^2 = |a_k(A_j + iB_j) - a_j(A_k + iB_k)|^2 = (a_k A_j - a_j A_k)^2 + (a_k B_j - a_j B_k)^2,$$

$j,k \in \{1,...,n\}$.

The member from the left from the (1) is equivalent to

$$\frac{A_1^2 + B_1^2}{a_1} + \frac{A_2^2 + B_2^2}{a_2} + ... + \frac{A_n^2 + B_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2 + (B_1 + B_2 + ... + B_n)^2}{a_1 + a_2 + ... + a_n} =$$
$$= \frac{1}{a_1 a_2 ... a_n (a_1 + a_2 + ... + a_n)}((a_2^2 a_3 a_4 ... a_n + a_2 a_3^3 a_4 ... a_n + ... + a_2 a_3 ... a_{n-1} a_n^2)(A_1^2 + B_1^2) +$$
$$+(a_1^2 a_3 a_4 ... a_n + a_1 a_3^2 a_4 ... a_n + ... + a_1 a_3 ... a_{n-1} a_n^2)(A_2^2 + B_2^2) + ... + (a_1^2 a_2 a_3 ... a_{n-1} +$$
$$+ a_1 a_2^2 a_3 ... a_{n-1} + ... + a_1 a_2 ... a_{n-2} a_{n-1}^2)(A_n^2 + B_n^2) - 2a_1 a_2 ... a_n((A_1 A_2 + A_1 A_3 + ... + A_1 A_n +$$
$$+ ... + A_{n-1} A_n) + (B_1 B_2 + B_1 B_3 + ... + B_1 B_n + ... + B_{n-1} B_n))) =$$
$$= \frac{1}{a_1 a_2 ... a_n (a_1 + a_2 + ... + a_n)} \cdot (a_3 a_4 ... a_n ((a_1 A_2 - a_2 A_1)^2 + (a_1 B_2 - a_2 B_1)^2)) +$$
$$+ a_2 a_4 a_5 ... a_n ((a_1 A_3 - a_3 A_1)^2 + (a_1 B_3 - a_3 B_1)^2) + ... + a_1 a_2 ... a_{n-2}((a_{n-1} A_n - a_n A_{n-1})^2 +$$
$$+(a_{n-1} B_n - a_n B_{n-1})^2)).$$

Thus we obtain,

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_1|^n}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} =$$
$$= \frac{A_1^2 + B_1^2}{a_1} + \frac{A_2^2 + B_2^2}{a_2} + ... + \frac{A_n^2 + B_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2 + (B_1 + B_2 + ... + B_n)^2}{a_1 + a_2 + ... + a_n} =$$
$$= \frac{1}{a_1 a_2 ... a_n (a_1 + a_2 + ... + a_n)}(a_3 a_4 ... a_n |a_1 N_2 - a_2 N_1|^2 + a_2 a_4 a_5 ... a_n |a_1 N_3 - a_3 N_1|^2 +$$
$$+ ... + a_1 a_2 ... a_{n-2} |a_{n-1} N_n - a_n N_{n-1}|^2).$$

□

Theorem 3. *(i) If $n \in N,\ n \leq 2,\ N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ are n gramian normal commutative operators and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} \geq \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} \tag{2}$$

(ii) Under the above conditions, if $a_1,\ a_2,\ ....,a_n \in (0, \infty)$ we also have,

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} \geq \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n},$$

with equality if and only if $a_iN_j = a_jN_i$, *for any* $i, j \in \{1, 2, ..., n\}$.

(iii) If we take in (i), $a_1 = a_2 = ... = a_n = 1$, *the inequality becomes,*

$$|N_1|^2 + |N_2|^2 + ... + |N_n|^2 \geq \frac{1}{n} \sum_{1 \leq i < j \leq n} |N_j - N_i|^2. \tag{3}$$

Proof. Will results immediately from the Theorem 2. The equality in (ii) holds from the proof of Theorem 2.

□

The following result is a particular case of Theorem 2.

Corollary 1. *If* $n \in N$, $n \leq 2$, $A_1, A_2, ..., A_n \in \mathcal{B}^*(\mathcal{H})$, *where* $\mathcal{H}$ *is a Loynes Z-space, are n gramian self-adjoint commutative operators and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n \neq 0$ *then,*

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_iA_j - a_jA_i)^2}{a_ia_j} \tag{4}$$

Using now the above corollary, we can obtain below a generalization of Remark 1 in (ii).

Corollary 2. *(i) If* $n \in N$, $n \leq 2$, $A_1, A_2, ..., A_n \in \mathcal{B}^*(\mathcal{H})$, *where* $\mathcal{H}$ *is a Loynes Z-space, are n gramian self-adjoint commutative operators and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n \neq 0$ *then,*

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} \geq \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_iA_j - a_jA_i)^2}{a_ia_j}$$

(ii) Under the above conditions, we have,

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} \geq \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n}$$

Consequence 1. *If* $n \in N$, $n \leq 2$, $N_1, N_2, ..., N_n \in \mathcal{B}(\mathcal{H})$, *where* $\mathcal{H}$ *is a Hilbert space, are n gramian normal commutative operators and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n \neq 0$ *then,*

$$\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + ... + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^2}{a_ia_j} \tag{5}$$

Consequence 2. *The points (i), (ii) and (iii) of Theorem 3 will be true also in case when* $\mathcal{H}$ *is a Hilbert space instead of Loynes space and the operators* $N_1, N_2, ..., N_n \in \mathcal{B}(\mathcal{H})$ *instead of* $\mathcal{B}^*(\mathcal{H})$.

Consequence 3. *Corollary 1 and Corollary 2 remain true also in case when $\mathcal{H}$ is a Hilbert space instead of Loynes space and the operators $A_1, A_2, ..., A_n$ are in $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}^*(\mathcal{H})$.*

The following result is an easy consequence of the Theorem 4 and Theorem 3 from [2].

Proposition 2. *Let $\mathcal{H}$ be a Hilbert space. For any operator $A_i \in \mathcal{B}(\mathcal{H})$, $i = \overline{1,n}$ we have:*

(i) If $\alpha_{ij} > 0$, $\beta_{ij} > 0, (\forall) i, j = \overline{1,n}$ then

$$|\sum_{i=1}^{n} A_i|^2 \leq \sum_{k=1}^{n}[1 + \sum_{j=k+1}^{n} \frac{\beta_{kj}}{\alpha_{kj}} + \sum_{j=1}^{k-1} \frac{\alpha_{kj}}{\beta_{kj}}]|A_k|^2;$$

(ii) If $\alpha_{ij} > 0$, $\beta_{ij} < 0$, $(\forall) i, j = \overline{1,n}$ then

$$|\sum_{i=1}^{n} A_i|^2 \geq \sum_{k=1}^{n}[1 + \sum_{j=k+1}^{n} \frac{\beta_{kj}}{\alpha_{kj}} + \sum_{j=1}^{k-1} \frac{\alpha_{kj}}{\beta_{kj}}]|A_k|^2;$$

Moreover, the equality in (i) and (ii) hold if and only if $\beta_{ij} A_i = \alpha_{ij} A_j$, $1 \leq i < j \leq n$.

Proof. We take $p_{ij} = \frac{\alpha_{ij}+\beta_{ij}}{\alpha_{ij}}$ and $q_{ij} = \frac{\alpha_{ij}+\beta_{ij}}{\beta_{ij}}$, where $1 \leq i < j \leq n$ and then from Theorem 4 we obtain the inequalities.

□

If we consider $\mathcal{H}$ a Loynes Z-space then the same result as the previous proposition is true.

The following lemma, remark and proposition are also true in Loynes Z-spaces as in Hilbert spaces, see [2].

Lemma 2. *For any $A_i \in \mathcal{B}^*(\mathcal{H}), i = 1, ..., n$ we have*

$$|\sum_{i=1}^{n} A_i|^2 - (\sum_{i=1}^{n} |A_i|)^2 = \sum_{1 \leq i < j \leq n} [|A_i + A_j|^2 - (|A_i| + |A_j|)^2].$$

Remark 2. *(a) For any A, $B \in \mathcal{B}^*(\mathcal{H})$ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have*

(i) $$|A - B|^2 + |(1 - p)A - B|^2 \leq p|A|^2 + q|B|^2$$

(ii) $$|A - B|^2 + |A - (1 - q)B|^2 \geq p|A|^2 + q|B|^2$$

with equality when $q = p = 2$ or $(1 - p)A = B$.

(b) For any A, $B \in \mathcal{B}^*(\mathcal{H})$ and every p, $q \in R$ with $\frac{1}{p} + \frac{1}{q} = 1$ if $p > 2$ we have:

(i) $$|A - B|^2 + |(1 - p)A - B|^2 \geq p|A|^2 + q|B|^2$$

$$|A - B|^2 + |A - (1 - q)B|^2 \leq p|A|^2 + q|B|^2.$$

Moreover the equality is if and only if $(1 - p)A = B$.

Proposition 3. *Let $\mathcal{H}$ be a Loynes Z- space. For any operator $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ we have:*

(i) If $\alpha_{ij} > 0$, $\beta_{ij} > 0 (\forall) i, j = \overline{1,n}$ then

$$|\sum_{i=1}^{n} A_i|^2 \leq \sum_{k=1}^{n} [1 + \sum_{j=k+1}^{n} \frac{\beta_{kj}}{\alpha_{kj}} + \sum_{j=1}^{k-1} \frac{\alpha_{kj}}{\beta_{kj}}] |A_k|^2;$$

(ii) If $\alpha_{ij} > 0$, $\beta_{ij} < 0$, $(\forall) i, j = \overline{1,n}$ then

$$|\sum_{i=1}^{n} A_i|^2 \geq \sum_{k=1}^{n} [1 + \sum_{j=k+1}^{n} \frac{\beta_{kj}}{\alpha_{kj}} + \sum_{j=1}^{k-1} \frac{\alpha_{kj}}{\beta_{kj}}] |A_k|^2;$$

Moreover, the equality in (i) and (ii) hold if and only if $\beta_{ij} A_i = \alpha_{ij} A_j$, $1 \leq i < j \leq n$.

The next result is a generalization of a result of [7].

Proposition 4. *Let $\mathcal{H}$ be a Loynes Z space. If $h_1, h_2, ..., h_n \in \mathcal{H}$ then*

$$q_p(\sum_{k=1}^{n} h_k) = \sum_{k=1}^{n} q_p(h_p) \tag{6}$$

is equivalent to

$$q_p(\sum_{k=1}^{n} a_k h_k) = \sum_{k=1}^{n} a_k q_p(h_p), \tag{7}$$

for any positive numbers $a_1, a_2, ..., a_n$.

Proof. As in [7] we can assume without loss of generality that $a_1 = \max_{k=1,2,...,n} a_k$. By (6) it results,

$$q_p(\sum_{k=1}^{n} a_k h_k) = q_p(a_1 \sum_{k=1}^{n} h_k - \sum_{k=1}^{n} (a_1 - a_k) h_k) \geq a_1 q_p(\sum_{k=1}^{n} h_k) - q_p(\sum_{k=1}^{n} (a_1 - a_k) h_k) \geq$$

$$\geq a_1 q_p(\sum_{k=1}^{n} h_k) - \sum_{k=1}^{n} (a_1 - a_k) q_p(h_k) = \sum_{k=1}^{n} a_k q_p(h_k).$$

Using now the triangle inequality, we have

$$q_p(\sum_{k=1}^{n} a_k h_k) \leq \sum_{k=1}^{n} q_p(a_k h_k) \leq \sum_{k=1}^{n} a_k q_p(h_k),$$

i.e. (7).

□

Proposition 5. *Let X be a space endowed with a family of seminorms which generates the topology of X. If $x_1, x_2, ..., x_n \in X$ then*

$$p(\sum_{k=1}^{n} x_k) = \sum_{k=1}^{n} p(x_p) \tag{8}$$

is equivalent to

$$p(\sum_{k=1}^{n} a_k x_k) = \sum_{k=1}^{n} a_k p(x_p), \tag{9}$$

for any positive numbers $a_1, a_2, ..., a_n$.

Using the Corollary 1, we can deduce an analogue of Theorem 5, see [9] for operators on Loynes Z- spaces.

Theorem 4. *If* $n \in N$ $n \geq 2$, $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ *are* n *gramian self-adjoint commutative and* $a_1, a_2, ..., a_n \in (0, \infty)$, *then*

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} \geq$$

$$\geq A_{k,l} + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j},$$

where

$$A_{k,l} = \max_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n.$$

Proof. The proof will be as in [9]. Using the inequality

$$\frac{A^2}{a} + \frac{B^2}{b} \geq \frac{(A + B)^2}{a + b},$$

from Remark 1 and Corollary 1, we have

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j} =$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\ m \neq k,l}^{n} \left(\frac{(a_m A_l - a_l A_m)^2}{a_m a_l} + \right.\right.$$

$$\left.\left. + \frac{(a_m A_k - a_k A_m)^2}{a_m a_k} \right)\right) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\ m \neq k,l}^{n} \left(\frac{(a_k A_l - \frac{a_l a_k}{a_m} A_m)^2}{\frac{a_l a_k^2}{a_m}} + \right.\right.$$

$$\left.\left. + \frac{(\frac{a_l a_k}{a_m} A_m - a_l A_k)^2}{\frac{a_k a_l^2}{a_m}} \right)\right) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j} \geq$$

$$\geq \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\ m \neq k,l}^{n} \frac{a_m (a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)} \right) +$$

$$+ \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j} =$$

$$= \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)}) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}.$$

□

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Department of Mathematics,"Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara, ROMANIA

E-mail address, L. Ciurdariu: cloredana43@yahoo.com