SOME INEQUALITIES IN C AND C^*-ALGEBRAS

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Abstract. In this paper we shall give some variants of some classical inequalities for gramian normal operators in pseudo-Hilbert spaces and Hilbert spaces and we shall present some consequences.

1. Introduction

We recall some equalities known from [2] and [9] for complex numbers.

Lemma 1. *If $z_1, z_2, ..., z_n,\ (n \geq 2)$ is a sequence of complex numbers, then*

$$\sum_{1\leq i<j\leq n} |z_i + z_j|^2 = (n-2)\sum_{k=1}^{n} |z_k|^2 + |\sum_{k=1}^{n} z_k|^2.$$

Lemma 2. *If $z_1, z_2, ..., z_n,\ (n \geq 2)$ is a sequence of complex numbers, then*

$$\sum_{1\leq i<j\leq n} |z_i - z_j|^2 = n\sum_{k=1}^{n} |z_k|^2 - |\sum_{k=1}^{n} z_k|^2.$$

As a generalization of the well-known identity,

$$(*)\qquad \frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} - \frac{|z_1+z_2|^2}{a_1+a_2} = \frac{|a_1 z_2 - a_2 z_1|^2}{a_1 a_2 (a_1+a_2)},$$

when a_1, a_2 are real numbers with $a_1,\ a_2 \neq 0,\ a_1 + a_2 \neq 0$, it was proved in [9] the following,

Theorem 1. *If $n \in N,\ n \leq 2,\ z_1, z_2, ..., z_n \in C$ are complex numbers and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + ... + \frac{|z_n|^2}{a_n} - \frac{|z_1+z_2+...+z_n|^2}{a_1+a_2+...+a_n} =$$

$$= \frac{1}{a_1+a_2+...+a_n}\sum_{1\leq i<j\leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j}.$$

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If we replace z_2 by $-z_2$ in (*) we also obtain

$$\frac{|z_1|^2}{a_1}+\frac{|z_2|^2}{a_2}-\frac{|z_1-z_2|^2}{a_1+a_2}=\frac{|a_1z_2+a_2z_1|^2}{a_1a_2(a_1+a_2)},$$

when a_1, a_2 are real numbers with $a_1,\ a_2\neq 0,\ a_1+a_2\neq 0$.

Taking now in equality (1), $a_i=1,\ (\forall)i\in\{1,2,...,n\}$ we notice that we shall obtain the equality from Lemma 2.

Also we need of a refinement of the classical Cauchy-Buniakowsky-Schwarz inequality which was proved in [1] by N. G. de Bruijin.

Theorem 2. *If $a_1,\ a_2,...,a_n$ is a sequence of real numbers and $z_1,\ z_2,...,z_n$ is a sequence of complex numbers then*

$$|\sum_{k=1}^{n}a_kz_k|^2\leq\frac{1}{2}\sum_{k=1}^{n}a_k^2[\sum_{k=1}^{n}|z_k|^2+|\sum_{k=1}^{n}z_k^2|],$$

equality holds when there is a α a complex number such that $\alpha^2\sum_{k=1}^{n}z_k^2\geq 0$ and $a_k=Re(\alpha z_k)$ for all k.

In section 3 we shall use the following results known from [3] and [4].
Let $\mathcal{H}$ be a Loynes Z- space, where Z is an admissible space, see [3], [6], [7].

Proposition 1. *For an operator $N\in\mathcal{C}^*(\mathcal{H})$ the following assertions are equivalent:*

(i): *N is gramian normal;*
(ii): *$[Nh,Nh]=[N^*h,N^*h],\ h\in\mathcal{H}$;*
(iii): *There exist two gramian self-adjoint commutative operators $A,B\in\mathcal{L}_h^*(\mathcal{H})$, such that $N=A+iB$.*

Theorem 3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Loynes Z–spaces and $N_1,N_2\in\mathcal{B}^*(\mathcal{H})$ two gramian normal operators on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. If there exists $T\in\mathcal{B}^*(\mathcal{H}_1,\mathcal{H}_2)$ which inverts N_1 and N_2 , that is $TN_1=N_2T$, then T also inverts the adjoints N_1^* and N_2^*, i.e. $TN_1^*=N_2^*T$.*

Theorem 4. *If $n\in N,\ n\leq 2,\ N_1,N_2,...,N_n\in\mathcal{B}^*(\mathcal{H})$ are n gramian normal commutative operators and $a_1,a_2,...,a_n\in R-\{0\}$ with $a_1+a_2+...+a_n\neq 0$ then,*

$$\frac{|N_1|^2}{a_1}+\frac{|N_2|^2}{a_2}+...+\frac{|N_n|^2}{a_n}-\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n}=$$

$$(1)\qquad =\frac{1}{a_1+a_2+...+a_n}\sum_{1\leq i<j\leq n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j},$$

where $|N|=(N^*N)^{\frac{1}{2}}$, it is known as modulus of N.

2. Some identities with complex numbers

First, we shall give a generalization of Lemma 1 for a sequence of complex numbers. Thus if we take in the following equality, $a_i = 1,\ i \in \{1, 2, ..., n\}$ we shall obtain exactly Lemma 1.

Theorem 5. *If $n \in N,\ n \leq 2,\ z_1, z_2, ..., z_n \in C$ are complex numbers and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|z_1 + z_2 + ... + z_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{2}{a_1 + a_2 + ... + a_n} \sum_{k=1}^{n} |z_k|^2$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j}. \tag{2}$$

Proof. We shall use the well-known polarization identity,

$$|z_i - z_j|^2 + |z_i + z_j|^2 = 2(|z_i|^2 + |z_j|^2),\ (\forall) i,\ j,\ 1 \leq i < j \leq n.$$

Let

$$S = \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j}.$$

Now, summing S with the right member of the identity (1), we shall have,

$$\frac{1}{\sum_{k=1}^{n} a_k} \Big(\sum_{1 \leq i < j \leq n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j} + \sum_{1 \leq i < j \leq n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \Big) =$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \sum_{1 \leq i < j \leq n} \frac{|a_i z_j|^2 + |a_j z_i|^2}{a_i a_j} =$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \Big(\sum_{j=2}^{n} \frac{|z_j|^2}{a_j} \sum_{i=1}^{j-1} \frac{|a_i|^2}{a_i} + \sum_{j=2}^{n} \frac{|a_j|^2}{a_j} \sum_{i=1}^{j-1} \frac{|z_i|^2}{a_i} \Big) =$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \Big(\sum_{j=2}^{n} \frac{|z_j|^2}{a_j} \sum_{i=1}^{j-1} a_i + \sum_{j=2}^{n} a_j \sum_{i=1}^{j-1} \frac{|z_i|^2}{a_i} \Big) =$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \Big(\sum_{j=2}^{n} \frac{|z_j|^2}{a_j} (a_1 + a_2 + ... + a_{j-1}) + \sum_{j=2}^{n} a_j \Big(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} + ... + \frac{|z_{j-1}|^2}{a_{j-1}} \Big) \Big) =$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \Big(\frac{|z_2|^2}{a_2} a_1 + \frac{|z_3|^2}{a_3} (a_1 + a_2) + ... + \frac{|z_n|^2}{a_n} (a_1 + a_2 + ... + a_{n-1}) + a_2 \frac{|z_1|^2}{a_1} +$$

$$+ a_3 \Big(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} \Big) + ... + a_n \Big(\frac{|z_1|^2}{a_1} + ... + \frac{|z_{n-1}|^2}{a_{n-1}} \Big) \Big)$$

$$= \frac{2}{\sum_{k=1}^{n} a_k} \Big(\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \sum_{k=1}^{n} |z_k|^2 \Big) =$$

Thus we have,

$$S = \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k} \sum_{k=1}^{n} |z_k|^2 + \frac{|\sum_{k=1}^{n} z_k|^2}{\sum_{k=1}^{n} a_k}.$$

□

Proposition 2. *Let* $z_1,\ z_2, ..., z_n,\ (n \geq 2)$ *be a sequence of complex numbers and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n > 0$. *Then*

$$\sum_{1\leq i<j\leq n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \leq \frac{n-4}{2}\sum_{k=1}^{n}|z_k|^2 + (\sum_{k=1}^{n} a_k)\cdot\sum_{k=1}^{n}\frac{|z_k|^2}{a_k} + \frac{n}{2}|\sum_{k=1}^{n} z_k^2|.$$

Proof. If we replace in inequality from Theorem 2, $a_k,\ k \in \{1,\ 2, ..., n\}$ by $\frac{1}{\sqrt{a_1+a_2+...+a_n}}$ then we have,

$$|\sum_{k=1}^{n}\frac{z_k}{\sqrt{a_1+a_2+...+a_n}}|^2 = \frac{|\sum_{k=1}^{n} z_k|^2}{\sum_{k=1}^{n} a_k} \leq \frac{1}{2}\sum_{k=1}^{n}\frac{1}{a_1+a_2+...+a_n}[\sum_{k=1}^{n}|z_k|^2 + |\sum_{k=1}^{n} z_k^2|]$$

or

$$\frac{|\sum_{k=1}^{n} z_k|^2}{\sum_{k=1}^{n} a_k} \leq \frac{n}{2\sum_{k=1}^{n} a_k}[\sum_{k=1}^{n}|z_k|^2 + |\sum_{k=1}^{n} z_k^2|].$$

Using now and Theorem 5, we find,

$$\frac{1}{a_1+a_2+...+a_n}\sum_{1\leq i<j\leq n}\frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \leq$$

$$\leq \frac{n}{2\sum_{k=1}^{n} a_k}[\sum_{k=1}^{n}|z_k|^2 + |\sum_{k=1}^{n} z_k^2|] + \sum_{k=1}^{n}\frac{|z_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k}\cdot\sum_{k=1}^{n}|z_k|^2 =$$

i.e.

$$\sum_{1\leq i<j\leq n}\frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \leq \frac{n}{2}[\sum_{k=1}^{n}|z_k|^2 + |\sum_{k=1}^{n} z_k^2|] + \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n}\frac{|z_k|^2}{a_k} - 2\sum_{k=1}^{n}|z_k|^2 =$$

$$= \frac{n-4}{2}\sum_{k=1}^{n}|z_k|^2 + \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n}\frac{|z_k|^2}{a_k} + \frac{n}{2}|\sum_{k=1}^{n} z_k^2|.$$

□

Remark 1. *If we take above in inequality* $a_k = 1,\ k \in \{1,\ 2, ..., n\}$ *then we obtain the inequality of Theorem 1 from* [2], *i.e.*

$$\sum_{1\leq i<j\leq n}|z_j + z_i|^2 \leq \frac{3n-4}{2}\sum_{k=1}^{n}|z_k|^2 + \frac{n}{2}|\sum_{k=1}^{n} z_k^2|.$$

Proposition 3. *Let* $z_1,\ z_2, ..., z_n,\ (n \geq 2)$ *be a sequence of complex numbers and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n > 0$. *Then*

$$\sum_{1\leq i<j\leq n}\frac{|a_i z_j - a_j z_i|^2}{a_i a_j} \geq \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n}\frac{|z_k|^2}{a_k} - \frac{n}{2}\sum_{k=1}^{n}|z_k|^2 - \frac{n}{2}|\sum_{k=1}^{n} z_k^2|.$$

Proof. Summing now the identities from Theorem 1 and 5 we find,

$$\frac{1}{\sum_{k=1}^{n} a_k} \sum_{1\le i<j\le n} \frac{|a_i z_j - a_j z_i|^2 + |a_i z_j + a_j z_i|^2}{a_i a_j} = 2\sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - \frac{2}{\sum_{k=1}^{n} a_k} \cdot \sum_{k=1}^{n} |z_k|^2$$

or

$$\sum_{1\le i<j\le n} \left(\frac{|a_i z_j - a_j z_i|^2}{a_i a_j} + \frac{|a_i z_j + a_j z_i|^2}{a_i a_j}\right) = 2\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - 2\sum_{k=1}^{n} |z_k|^2.$$

Applying now Proposition 2 we deduce that,

$$\sum_{1\le i<j\le n} \frac{|a_i z_j - a_j z_i|^2}{a_i a_j} = 2\sum_{k=1}^{n} a_k \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} - 2\sum_{k=1}^{n} |z_k|^2 - \sum_{1\le i<j\le n} \frac{|a_i z_j + a_j z_i|^2}{a_i a_j} \le$$

$$\le \frac{n-4}{2} \sum_{k=1}^{n} |z_k|^2 + \left(\sum_{k=1}^{n} a_k\right) \cdot \sum_{k=1}^{n} \frac{|z_k|^2}{a_k} + \frac{n}{2} \left|\sum_{k=1}^{n} z_k^2\right|,$$

which gives desired conclusion inequality.

□

Remark 2. *If we take above in inequality $a_k = 1,\ k \in \{1,\ 2, ..., n\}$ then we obtain the inequality of Theorem 2 from* [2], *i.e.*

$$\sum_{1\le i<j\le n} |z_j - z_i|^2 \ge \frac{n}{2}\left(\sum_{k=1}^{n} |z_k|^2 - \left|\sum_{k=1}^{n} z_k^2\right|\right).$$

3. The main results

In this section we shall denote by $\mathcal{B}^*(\mathcal{H})$ the space of linear, gramian bounded operators which admit gramian adjoint, see [6], [7], if $\mathcal{H}$ is a pseudo-Hilbert space and by $\mathcal{B}(\mathcal{H})$ the space of linear bounded operators if $\mathcal{H}$ is a Hilbert space.

Now, we shall generalize Theorem 5 for a class of particular operators on Loynes spaces.

Let $\mathcal{H}$ be a Loynes Z-space.

In the proof of the below theorem we need of the following equality:

$$|N_1 - N_2|^2 + |N_1 + N_2|^2 = 2|N_1|^2 + 2|N_2|^2, \tag{3}$$

where $|N_1| = (N_1^* N_1)^{\frac{1}{2}}$ means the modulus of N_1 and $N_1,\ N_2 \in \mathcal{B}^*(\mathcal{H})$ are arbitrary operators.

The above equality will be easy verified by calculus.

Theorem 6. *If $n \in N,\ n \le 2,\ N_1,\ N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H})$ are n gramian normal operators which commute as pairs and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{|N_1 + N_2 + ... + N_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{2}{a_1 + a_2 + ... + a_n} \sum_{k=1}^{n} |N_k|^2$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1\le i<j\le n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j}. \tag{4}$$

Proof. Using now identity (1) of Theorem 4 and equality (3), as in the proof of Theorem 5, we obtain,

$$\frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}=\frac{1}{\sum_{k=1}^n a_k}(\sum_{1\le i<j\le n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j}+$$

$$+\sum_{1\le i<j\le n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j})-\frac{1}{\sum_{k=1}^n a_k}\sum_{1\le i<j\le n}\frac{|a_iN_j-a_jN_i|^2}{a_ia_j}=$$

$$=\frac{2}{\sum_{k=1}^n a_k}(\sum_{k=1}^n a_k\cdot\sum_{k=1}^n\frac{|N_k|^2}{a_k}-\sum_{k=1}^n|N_k|^2)-(\sum_{k=1}^n\frac{|N_k|^2}{a_k}-\frac{|\sum_{k=1}^n N_k|^2}{\sum_{k=1}^n a_k})$$

□

Remark 3. *If $n\in N$, $n\le 2$, $N_1,\ N_2,...,N_n\in\mathcal{B}^*(\mathcal{H})$ are n gramian normal operators which commute as pairs and $a_1,a_2,...,a_n\in R-\{0\}$ with $a_1+a_2+...+a_n>0$ then,*

(5)

$$\frac{|N_1+N_2+...+N_n|^2}{a_1+a_2+...+a_n}+\sum_{k=1}^n\frac{|N_k|^2}{a_k}\ge\frac{1}{a_1+a_2+...+a_n}\sum_{1\le i<j\le n}\frac{|a_iN_j+a_jN_i|^2}{a_ia_j}.$$

Remark 4. *If $\mathcal{H}$ is a Hilbert space instead of Loynes Z-space, $n\in N$, $n\le 2$, $N_1,\ N_2,...,N_n\in\mathcal{B}(\mathcal{H})$ are n gramian normal operators which commute as pairs and $a_1,a_2,...,a_n\in R-\{0\}$ with $a_1+a_2+...+a_n>0$ then, the equality (4) from Theorem 6 and inequality (5) from Remark 3 remain true.*

Now using the proof of Theorem 4, see [8], we shall give for this particular class of seminorms, q_p on pseudo-Hilbert spaces $\mathcal{H}$, see [3], the following result, see [8] for real positive numbers. We can notice that the particular form of q_p doesn′t matter and that we can consider in the proof of theorem not only that $d_n\ge d_3$, even $d_n\ge d_{n-1}$ to obtain other refinements, where we considered

$$d_n=\frac{q_p(h_1)^{r+1}}{a_1^r}+...+\frac{q_p(h_n)^{r+1}}{a_n^r}-\frac{q_p(h_1+...+h_n)^{r+1}}{(a_1+...+a_n)^r}.$$

Theorem 7. *For $a_k,\ h_k\in\mathcal{H}$ with $g_p(h_k)>0,\ r\ge 1,\ k\in\{1,2,...,n\},\ n\ge 3,\ n\in N$ the following inequality takes place:*

$$\sum_{k=1}^n\frac{q_p(h_k)^{r+1}}{a_k^r}\ge\frac{(\sum_{k=1}^n q_p(h_k))^{r+1}}{(\sum_{k=1}^n a_k)^r}+$$

$$+\max_{1\le i<j<k\le n}(\frac{q_p(h_i)^{r+1}}{a_i^r}+\frac{q_p(h_j)^{r+1}}{a_j^r}+\frac{q_p(h_k)^{r+1}}{a_i^k}-\frac{q_p(h_i+h_j+h_k)^{r+1}}{(a_i+a_j+a_k)^r}).$$

We recall further the following result known from [4] which will be used below.

Theorem 8. *If $n \in N$ $n \geq 2$, $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ gramian self-adjoint operators which commute as pairs and $a_1, a_2, ..., a_n \in (0, \infty)$, then*

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} \geq$$

$$\geq A_{k,l} + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j},$$

where

$$A_{k,l} = \max_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n.$$

Theorem 9. *If $n \in N$ $n \geq 2$, $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$ gramian self-adjoint operators which commute as pairs and $a_1, a_2, ..., a_n \in (0, \infty)$, then*

$$\frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j + a_j A_i)^2}{a_i a_j} \geq$$

$$\geq \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{A_k^2}{a_k} -$$

$$-2[\frac{A_k^2}{a_k} + \frac{A_l^2}{a_l} + \frac{a_k + a_l}{\sum_{i=1}^n a_i} \sum_{r=1, r \neq k,l}^{n} \frac{A_r^2}{a_r} + \frac{1}{\sum_{i=1}^n a_i} \sum_{t=2, t \neq k,l}^{n-1} A_t^2] + A_{k,l},$$

where

$$A_{k,l} = \max_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n.$$

Proof. We shall use the inequality from Theorem 8,

$$\sum_{k=1}^{n} \frac{A_k^2}{a_k} - \frac{(\sum_{k=1}^n A_k)^2}{\sum_{k=1}^n a_k} \geq A_{k,l} + \frac{1}{\sum_{k=1}^n a_k} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j} =$$

$$= A_{k,l} + \frac{2}{\sum_{k=1}^n a_k} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j)^2 + (a_j A_i)^2}{a_i a_j} -$$

$$-\frac{1}{\sum_{k=1}^n a_k} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j + a_j A_i)^2}{a_i a_j} = A_{k,l} +$$

$$+\frac{2}{\sum_{k=1}^n a_k} (\sum_{1 \leq i < j \leq n} \frac{(a_i A_j)^2 + (a_j A_i)^2}{a_i a_j} - \frac{(a_k A_l)^2 + (a_l A_k)^2}{a_k a_l} -$$

$$- \sum_{m=1,\ m \neq k,l}^{n} (\frac{(a_m A_l)^2 + (a_l A_m)^2}{a_m a_l} + \frac{(a_m A_k)^2 + (a_k A_m)^2}{a_m a_k})) -$$

$$-\frac{1}{\sum_{k=1}^n a_k} \sum_{1 \leq i < j \leq n,\ i,j \neq k,l} \frac{(a_i A_j + a_j A_i)^2}{a_i a_j} =$$

$$= A_{k,l} + \frac{2}{\sum_{k=1}^n a_k} (\sum_{j=2}^{n} \frac{A_j^2}{a_j} \sum_{i=1}^{j-1} a_i + \sum_{j=2}^{n} a_j \sum_{i=1}^{j-1} \frac{A_i^2}{a_i}) - \frac{2}{\sum_{i=1}^n a_i} \frac{(a_k A_l)^2 + (a_l A_k)^2}{a_k a_l} -$$

$$-\frac{2}{\sum_{i=1}^{n} a_i}((\frac{A_l^2}{a_l}+\frac{A_k^2}{a_k})\cdot\sum_{m=1,\ m\neq k,l}^{n} a_m+(a_l+a_k)\cdot\sum_{m=1,\ m\neq k,l}\frac{A_m^2}{a_m})-$$

$$-\frac{1}{\sum_{k=1}^{n} a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}.$$

By calculus the last term of previous inequality become

$$A_{k,l}-\frac{1}{\sum_{k=1}^{n} a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n} a_i}(a_1\sum_{i=2,i\neq k,l}^{n}\frac{A_i^2}{a_i}+$$

$$+a_2\sum_{i=3,i\neq k,l}^{n}\frac{A_i^2}{a_i}+...+a_{k-1}\sum_{i=k+1,i\neq l}^{n}\frac{A_i^2}{a_i}-a_k\sum_{i=1}^{k-1}\frac{A_i^2}{a_i}+a_{k+1}\sum_{i=k+2,i\neq l}^{n}\frac{A_i^2}{a_i}+...+$$

$$+a_{l-1}\sum_{i=l+1}^{n}\frac{A_i^2}{a_i}-a_l\sum_{i=1}^{l-1}\frac{A_i^2}{a_i}+...+a_{n-1}\frac{A_n^2}{a_n}+a_2\frac{A_1^2}{a_1}+a_3(\frac{A_1^2}{a_1}+\frac{A_2^2}{a_2})+...+a_n\sum_{i=1,i\neq k,l}^{n-1}\frac{A_i^2}{a_i})=$$

$$=A_{k,l}-\frac{1}{\sum_{k=1}^{n} a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n} a_i}(\sum_{t=1,t\neq k,l}^{n-1} a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}-$$

$$-a_k\sum_{i=1}^{k-1}\frac{A_i^2}{a_i}-a_l\sum_{i=1}^{l-1}\frac{A_i^2}{a_i}+\sum_{p=2}^{n} a_p\sum_{r=1,r\neq k,l}^{p-1}\frac{A_r^2}{a_r})=A_{k,l}-$$

$$-\frac{1}{\sum_{k=1}^{n} a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n} a_i}(\sum_{t=1,t\neq k,l}^{n-1} a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}+$$

$$+\sum_{t=2,t\neq k,l}^{n} a_t\sum_{r=1,r\neq k,l}^{t-1}\frac{A_r^2}{a_r})$$

Because,

$$\sum_{t=1,t\neq k,l}^{n-1} a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}+\sum_{t=2,t\neq k,l}^{n} a_t\sum_{r=1,r\neq k,l}^{t-1}\frac{A_r^2}{a_r}=\sum_{t=2,t\neq k,l}^{n-1} a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}+$$

$$+\sum_{t=2,t\neq k,l}^{n-1} a_t\sum_{r=1,r\neq k,l}^{t-1}\frac{A_r^2}{a_r}+a_1\sum_{r=2,r\neq k,l}^{n}\frac{A_r^2}{a_r}+a_n\sum_{r=2,r\neq k,l}^{n-1}\frac{A_r^2}{a_r}=\sum_{t=1,t\neq k,l}^{n} a_t\sum_{r=1,r\neq k,l}^{n}\frac{A_r^2}{a_r}-$$

$$-\sum_{t=2,t\neq k,l}^{n-1} A_t^2=\sum_{t=1}^{n} a_t\sum_{r=1,r\neq k,l}^{n}\frac{A_r^2}{a_r}-(a_k+a_l)\sum_{r=1,r\neq k,l}^{n}\frac{A_r^2}{a_r}-\sum_{t=2,t\neq k,l}^{n-1} A_t^2,$$

we shall obtain the inequality.

□

Consequence 1. *If $n\in N$ $n\geq 2$, $\mathcal{H}$ being now a Hilbert space, $A_i\in\mathcal{B}(\mathcal{H}),\ i=\overline{1,n}$ gramian self-adjoint operators which commute as pairs and $a_1,a_2,...,a_n\in(0,\infty)$, then the same inequality as in Theorem 9 is satisfied.*

The inequality of N. G. de Bruijn can have the following form.

Theorem 10. *If $a_1,\ a_2, ..., a_n$ is a sequence of real numbers and $N_1,\ N_2, ..., N_n$ is a sequence of gramian normal commuting operators in $\mathcal{B}^*(\mathcal{H})$ such that $\sum_{k=1}^n a_k N_k \geq 0$ then*

$$|\sum_{k=1}^n a_k N_k|^2 \leq \frac{1}{2}\sum_{k=1}^n a_k^2[\sum_{k=1}^n |N_k|^2 + |\sum_{k=1}^n N_k^2|].$$

where $|N|$ means the modulus of the operator N.

Proof. The proof will be as in [5]. Using that $\sum_{k=1}^n a_k N_k \geq 0$ and the definition of modulus of an operator we obtain,

$$|\sum_{k=1}^n a_k N_k|^2 = (\sum_{k=1}^n a_k N_k)^2,$$

where $N_k = A_k + iB_k,\ k = \overline{1,n}$.

Again from $\sum_{k=1}^n a_k N_k \geq 0$ it results $\sum_{k=1}^n a_k N_k = \sum_{k=1}^n a_k N_k^*$ or

$$\sum_{k=1}^n a_k B_k = 0.$$

Thus

$$|\sum_{k=1}^n a_k N_k|^2 = |\sum_{k=1}^n a_k A_k + i\sum_{k=1}^n a_k B_k|^2 = (\sum_{k=1}^n a_k A_k)^2 + (\sum_{k=1}^n a_k B_k)^2 =$$

$$= (\sum_{k=1}^n a_k A_k)^2 \leq \sum_{k=1}^n a_k^2 \sum_{k=1}^n A_k^2,$$

the last inequality being the Cauchy-Schwarz inequality for gramian self-adjoint operators which commute as pairs. A proof for this last inequality is for example by induction, using that for gramian self-adjoint commuting operators $A_k,\ A_l$ and $a_k,\ a_l$ real numbers, $k, l \in \{1, ..., n\}$, we have $(a_k A_l - a_l A_k)^2 \geq 0$. Now taking into account that for all $k \in \{1, ..., n\}$, $2A_k^2 = |N_k|^2 + C_k$, where $C_k = A_k^2 - B_k^2$, will have $N_k^2 = C_k + 2iA_k B_k$ and then,

$$|\sum_{k=1}^n a_k N_k|^2 \leq \frac{1}{2}\sum_{k=1}^n a_k^2[\sum_{k=1}^n |N_k|^2 + \sum_{k=1}^n C_k].$$

Because

$$\sum_{k=1}^n C_k \leq |\sum_{k=1}^n N_k^2|$$

we have the desired inequality.

□

As an analogue of the Proposition 2, see section 2, we can state the following result for operators:

Proposition 4. *Let $N_1,\ N_2, ..., N_n,\ (n \geq 2)$ be a sequence of gramian normal operators in $\mathcal{B}^*(\mathcal{H})$ commuting as pairs such that $\sum_{k=1}^n a_k N_k \geq 0$ and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n > 0$. Then*

$$\sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^2}{a_i a_j} \leq \frac{n-4}{2}\sum_{k=1}^n |N_k|^2 + (\sum_{k=1}^n a_k) \cdot \sum_{k=1}^n \frac{|N_k|^2}{a_k} + \frac{n}{2}|\sum_{k=1}^n N_k^2|.$$

The idea of the proof will be as in the Proposition 2.

Remark 5. *If we take in the above inequality* $a_k = 1,\ k \in \{1,\ 2, ..., n\}$ *then we obtain*

$$\sum_{1\le i<j\le n} |N_j + N_i|^2 \le \frac{3n-4}{2}\sum_{k=1}^{n}|N_k|^2 + \frac{n}{2}|\sum_{k=1}^{n} N_k^2|.$$

Proposition 5. *Let* $N_1,\ N_2, ..., N_n,\ (n \ge 2)$ *be a sequence of gramian normal commuting operators in* $\mathcal{B}^*(\mathcal{H})$ *such that* $\sum_{k=1}^{n} a_k N_k \ge 0$ *and* $a_1, a_2, ..., a_n \in R-\{0\}$ *with* $a_1 + a_2 + ... + a_n > 0$. *Then*

$$\sum_{1\le i<j\le n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j} \ge \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{|N_k|^2}{a_k} - \frac{n}{2}\sum_{k=1}^{n}|N_k|^2 - \frac{n}{2}|\sum_{k=1}^{n} N_k^2|.$$

The idea of the proof will be as in the Proposition 3.

Remark 6. *If we take above, in inequality* $a_k = 1,\ k \in \{1,\ 2, ..., n\}$ *then we obtain*

$$\sum_{1\le i<j\le n} |N_j - N_i|^2 \ge \frac{n}{2}(\sum_{k=1}^{n}|N_k|^2 - |\sum_{k=1}^{n} N_k^2|).$$

Consequence 2. *If we consider above in the Propositions 4 and 5, Remark 5 and 6* $\mathcal{H}$ *as Hilbert space instead of pseudo-Hilbert space and* $\mathcal{B}(\mathcal{H})$ *instead of* $\mathcal{B}^*(\mathcal{H})$, *then the conclusions remain true.*

We can also think to use in the proves the continuous functional calculus for normal bounded operators.

4. An identity in C^*-algebras

It is known that $\mathcal{B}^*(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ are particular C^*-algebras, see [7], [3], [10]. As in case of pseudo-Hilbert spaces, we can also deduce by calculus in C^*-algebras the following equality:

$$|b_1 - b_2|^2 + |b_1 + b_2|^2 = 2|b_1|^2 + 2|b_2|^2$$

where $|b_1| = (b_1^* b_1)^{\frac{1}{2}}$ means the modulus of b_1 and $b_1,\ b_2 \in \mathcal{A}$ are arbitrary elements in a C^*-algebra $\mathcal{A}$. Then the following theorems will be proven as in pseudo-Hilbert spaces using the similar theorems, see [10].

Theorem 11. *If* $n \in N,\ n \le 2,\ b_1,\ b_2, ..., b_n \in \mathcal{A}$ *are n normal elements which commute as pairs and* $a_1, a_2, ..., a_n \in R - \{0\}$ *with* $a_1 + a_2 + ... + a_n \ne 0$ *then,*

$$\frac{|b_1 + b_2 + ... + b_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|b_k|^2}{a_k} - \frac{2}{a_1 + a_2 + ... + a_n}\sum_{k=1}^{n}|b_k|^2$$

$$= \frac{1}{a_1 + a_2 + ... + a_n}\sum_{1\le i<j\le n} \frac{|a_i b_j + a_j b_i|^2}{a_i a_j}.$$

Using the above result, it easily results the following:

Remark 7. *If $n \in N$, $n \leq 2$, $b_1, b_2, ..., b_n \in \mathcal{A}$ are n normal elements which commute as pairs and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n > 0$ then,*

$$\frac{|b_1 + b_2 + ... + b_n|^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{|b_k|^2}{a_k} \geq \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{|a_i b_j + a_j b_i|^2}{a_i a_j}.$$

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