REFINEMENTS OF CHOI–DAVIS–JENSEN'S INEQUALITY

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ABSTRACT. In this paper, we review some results concerning with Jensen's inequality, some equivalent conditions to the operator convexity, inequalities involving eigenvalues and present some refinements of the Choi–Davis–Jensen inequality $f(\Phi(A)) \leq \Phi(f(A))$ for strictly positive maps.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ stand for the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ the order relation $A \leq B$ means that $\langle A\xi, \xi \rangle \leq \langle B\xi, \xi \rangle$ $(\xi \in \mathscr{H})$. In particular, if $0 \leq A$, then A is called *positive*. If a positive operator A is invertible, then we say that it is *strictly positive* and write $0 < A$. Every positive operator B has a unique positive square root $B^{1/2}$, in particular, the absolute value of $A \in \mathbb{B}(\mathscr{H})$ is defined to be $|A| = (A^*A)^{1/2}$. Throughout the paper any C^*-algebra $\mathscr{A}$ is regarded as a closed $*$-subalgebra of $\mathbb{B}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$. We also use the same notation I for denoting the identity of C^*-algebras in consideration.

A continuous real function f defined on an interval J is called *operator convex* if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all $0 \leq \lambda \leq 1$ and all self-adjoint operators A, B with spectra in J. A function f is called *operator concave* if $-f$ is operator convex.

A linear map $\Phi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras is said to be *positive* if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is *unital* if Φ preserves the identity. The linear map Φ is called *strictly positive* if $\Phi(A)$ is strictly positive whenever A is strictly positive. It can be easily seen that a positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$. For a comprehensive account on positive linear maps see [9].

The notion of Hilbert C^*-module is an extension of that of Hilbert space, where the inner product takes its values in a C^*-algebra. If $(\mathscr{X}, \langle \cdot, \cdot \rangle)$ is a Hilbert C^*-module over a

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C^*-algebra, then $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ defines a complete norm on $\mathscr{X}$, where the latter norm denotes that in the C^*-algebra. Any Hilbert space can be regarded as a Hilbert $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a,b\rangle = a^*b$ $(a,b\in\mathscr{A})$.
The set of all maps T on a Hilbert C^*-module $\mathscr{X}$ such that there is a map T^* on $\mathscr{X}$ with the property $\langle T(x),y\rangle = \langle x,T^*(y)\rangle$ $(x,y\in\mathscr{X})$ is denoted by $\mathcal{L}(\mathscr{X})$. This space is in fact a unital C^*-algebra in a natural fashion. For every $x\in\mathscr{X}$ we define the absolute value of x as the unique positive square root of $\langle x,x\rangle$, that is, $|x| = \langle x,x\rangle^{\frac{1}{2}}$. For any $x,y\in\mathscr{X}$ the operator $x\otimes y$ on $\mathscr{X}$ is defined by $(x\otimes y)(z) = x\langle y,z\rangle$ $(z\in\mathscr{X})$.
We refer the reader to [34] for undefined notions on C^*-algebra theory and to [29] for more information on Hilbert C^*-modules.

In section 2, we review some results related to Jensen's operator inequality. Section 3 deals with some inequalities involving eigenvalues. Section 4 is devoted to survey some inequalities related to $f(A+B)$ and $f(A)+f(B)$. A discussion of equivalent conditions to the operator convexity is given in Section 5. We present some refinements of the Choi–Davis–Jensen inequality $f(\Phi(A))\le\Phi(f(A))$ for strictly positive maps in the last section.

2. Jensen's operator inequality

The classical Jensen inequality states that if f is a convex function on an interval J then for elements $x_1,\dots,x_n\in J$, we have

$$f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i f(x_i), \tag{2.1}$$

where $t_1,\dots,t_n$ are positive real number with $\sum_{i=1}^n t_i = 1$.

To find an operator version of Jensen's inequality, let us consider the matrices $A = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$ and $x = \begin{pmatrix} \sqrt{t_1} \\ \vdots \\ \sqrt{t_n} \end{pmatrix}$. Then inequality (2.1) can be stated as

$$f(\langle Ax,x\rangle) \le \langle f(A)x,x\rangle.$$

Using this approach, Mond and Pečarić [32, 33] proved that if f is a convex function on an interval J and A is a self-adjoint operator on a Hilbert space $\mathscr{H}$ with spectrum in J, then

$$f(\langle Ax,x\rangle) \le \langle f(A)x,x\rangle, \tag{2.2}$$

for every unit vector $x \in \mathscr{H}$. If we put $A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in (2.2), we obtain a multiple operator version of the Jensen inequality as follows:

$$f\left(\sum_{i=1}^{n} \langle A_i x_i, x_i \rangle\right) \leq \sum_{i=1}^{n} \langle f(A_i) x_i, x_i \rangle,$$

where $\sum_{i=1}^{n} \|x_i\|^2 = 1$.

Another approach to an operator version of (2.1) can be obtained as follows:

If $A = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$ and $V = \begin{pmatrix} \sqrt{t_1} & \dots, 0 \\ \vdots & \vdots \\ \sqrt{t_n} & \dots, 0 \end{pmatrix}$, then we can express (2.1) as

$$f(V^* A V) \leq V^* f(A) V.$$

Similar to inequality (2.2), it can be proved that if $\mathscr{A}$ is a C^*-algebra and φ is a state on $\mathscr{A}$, then for every convex function f, the inequality

$$f(\varphi(a)) \leq \varphi(f(a)), \tag{2.3}$$

holds, for each $a \in \mathscr{A}$. But it is not generally true when the state φ is replaced by an arbitrary positive linear map between C^*-algebras. However, by using Stinespring decomposition theorem, Davis [22] and Choi [20] showed that if Φ is a normalized positive linear map on $B(\mathscr{H})$ and if f is an operator convex function on an interval J, then the so-called Choi–Davis–Jensen inequality

$$f(\Phi(A)) \leq \Phi(f(A)), \tag{2.4}$$

holds for every self-adjoint operator A on $\mathscr{H}$ whose spectrum is contained in J. Ando [1] gave an alternative proof for this inequality by using the integral representation of operator convex functions. The equivalence of the Choi–Davis–Jensen inequality and the operator convexity of f is proved by J.I. Fujii and M. Fujii [23].

A number of mathematicians investigated some different types of inequality (2.3), when f is not necessarily operator convex.

In [3], Antezana, Massey and Stojanoff proved the following interesting result.

Theorem 2.1. *Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a positive unital map between unital C^*-algebras $\mathscr{A}, \mathscr{B}$ and f is a convex function. If $a \in \mathscr{A}$ such that $\Phi(f(a))$ and $\Phi(a)$ commute, then*

$$f(\Phi(a)) \leq \Phi(f(a)).$$

They also proved this theorem for the multi-variable case.

The following Jensen type inequality of Mercer's type was proved in [31].

Theorem 2.2. *Let $\Phi_1, \ldots, \Phi_n$ be positive linear maps from $\mathbb{B}(\mathscr{H})$ into $\mathbb{B}(\mathscr{K})$ such that $\sum_{i=1}^{n} \Phi_i(I) = I$, f be a convex continuous function on an interval $[m, M]$ and $A_1, \ldots, A_n$ be self-adjoint operators on $\mathscr{H}$ with spectra in $[m, M]$. Then*

$$f(mI + MI - \sum_{i=1}^{n} \Phi_i(A_i)) \leq f(m)I + f(M)I - \sum_{i=1}^{n} \Phi_i(f(A_i)).$$

3. Inequalities involving eigenvalues

Before expressing the versions of Jensen's inequality involving eigenvalues we recall some fact in this area.

Let A be a hermitian matrix, we enumerate its eigenvalues as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and for an arbitrary matrix A, we use $s_1(A) \geq \cdots \geq s_n(A)$ as the singular values of A (i.e. the eigenvalues of $|A|$). By Weyl's monotonicity theorem , $A \leq B$ implies $\lambda_j(A) \leq \lambda_j(B)$ for all $1 \leq j \leq n$. This set of inequalities is equivalent to $A \leq U^*BU$ for some unitary matrix U. If for all $1 \leq k \leq n$, we have $\sum_{j=1}^{k} \lambda_j(A) \leq \sum_{j=1}^{k} \lambda_j(B)$, we say that $\lambda(A)$ is weakly majorised by $\lambda(B)$ and show it by $\lambda(A) \prec_\omega \lambda(B)$. The Fan dominance theorem says that $s(A) \prec_\omega s(B)$ if and only if $|||A||| \leq |||B|||$ for every unitarily invariant norm. We refer the interested reader to [8] for more information about majorization.

If the condition of operator convexity is replaced by convexity, almost all of the inequalities in the previous section fail to be valid. There are some attempts by using the majorization concept to find some weaker inequalities, when f is a convex function. Throughout this section, all of the operators are in a finite dimensional Hilbert space.

As a result of (2.2), Aujla and Silva [7] showed that if f is a convex function then

$$\lambda(f(\alpha A + (1-\alpha)B)) \prec_\omega \lambda(\alpha f(A) + (1-\alpha)f(B)),$$

and if further $f(0) \leq 0$ then

$$\lambda(f(X^*AX)) \prec_\omega \lambda(X^*f(A)X),$$

for every contraction X.

By taking $f(t) = -\log t$, they show that the above inequalities may not be true for symmetric norms, unless we assume that f is nonnegative.

In addition if f is increasing or decreasing, then the following stronger results hold:

$$\lambda_j(f(\alpha A + (1-\alpha)B)) \leq \lambda_j(\alpha f(A) + (1-\alpha)f(B)),$$

and

$$\lambda_j(f(X^*AX)) \leq \lambda_j(X^*f(A)X).$$

The last result was proved by Bourin [12] with a different technique using compression. Also he mentioned that if $\sum_{j=1}^n X_j^*X_j = I$ then

$$\lambda_j(f(\sum_{j=1}^n X_j^*A_jX_j)) \leq \lambda_j(\sum_{j=1}^n X_j^*f(A_j)X_j).$$

If f is convex, we can write it as $f(x) = g(x) - \lambda x$ for some convex monotone function g and some scalar λ. Applying the recent result to g, one can get the well-known results of Brown–Kosaki [19] and Hansen–Pedersen [26], which state, respectively, that

- If Z is contraction and $f(0) \leq 0$, then

$$\mathrm{Tr}(f(Z^*AZ)) \leq \mathrm{Tr}(Z^*f(A)Z)$$

- If $\sum_{j=1}^n Z_j^*Z_j = I$ then

$$\mathrm{Tr}(f(\sum_{j=1}^n Z_j^*A_jZ_j)) \leq \mathrm{Tr}(\sum_{j=1}^n Z_j^*f(A_j)Z_j)$$

Further, by regarding the convex function $f(t) = |t|$, Bourin noted that his result fails to be true when f is not increasing. But he used the operator version

$$|A+B| \leq U|A|U^* + V|B|V^*,$$

of the triangle inequality, where U and V are unitaries, to show that for even convex function f,

$$f\left(\frac{A+B}{2}\right) \leq \frac{Uf(A)U^* + Vf(B)V^*}{2}.$$

In [13], these inequalities were improved as follows:

- If f is convex and A is self-adjoint, then for each contraction Z there exist unitaries U and V such that

$$X \leq \frac{UYU^* + VYV^*}{2},$$

  where $X = f(Z^*AZ)$ and $Y = Z^*f(A)Z$.

- If f is convex and $A_1, \dots, A_n$ are self-adjoint and $\sum_{i=1}^n Z_i^* Z_i = I$ there exist unitaries U and V such that
$$X \leq \frac{UYU^* + VYV^*}{2},$$
where $X = f(\sum_{i=1}^n Z_i^* A_i Z_i)$ and $Y = \sum_{i=1}^n Z_i^* f(A_i) Z_i$.

Recently, the following theorem was proved by Antezana, Massey and Stojanoff [3]. See also [6, 7, 36]

Theorem 3.1. *Let Φ be a unital positive map from $\mathscr{A}$ to the algebra $\mathscr{M}_n$ of all $n \times n$ matrices, $a \in \mathscr{A}$ and f is a convex function, whose domain is an interval containing the spectrum of a. Then*
$$f(\Phi(a)) \prec_\omega \Phi(f(a)).$$
If in addition f is monotone then
$$\lambda_i(f(\Phi(a))) \leq \lambda_i(\Phi(f(a)))$$
for every $1 \leq i \leq n$.

4. Inequalities related to $f(A+B)$ and $f(A)+f(B)$

Regarding a question raised by Bhatia and Kittaneh [10], Ando and Zhan [2] proved that $|||f(A)+f(B)||| \geq |||f(A+B)|||$ for any positive semi-definite matrices A, B, any unitarily invariant norm $|||\cdot|||$ on $\mathscr{M}_n$, and any nonnegative operator monotone function f on $[0,\infty)$. They also showed that $|||f(A)+f(B)||| \leq |||f(A+B)|||$ for a nonnegative increasing function f on $[0,\infty)$ with $f(0)=0$ and $f(\infty)=\infty$, whose inverse is operator monotone. The versions of the results above for n-tuples were given by Bhatia and Kittaneh [11]. Later, Kosem [28] proved the following general result:

- If $f:[0,\infty) \to [0,\infty)$ is a convex function with $f(0)=0$, then for any positive operators $A_1, \cdots, A_n$ and any unitarily invariant norm $|||\cdot|||$ on $\mathbb{B}(\mathscr{H})$
$$\left|\left|\left|\sum_{j=1}^n f(A_j)\right|\right|\right| \leq \left|\left|\left| f\left(\sum_{j=1}^n A_j\right)\right|\right|\right|.$$

Also, Bourin and Uchiyama extended the result of Ando–Zhan as follows [18] (see also [6]).

• If $f : [0, \infty) \to [0, \infty)$ is a concave function, then for any positive operators $A_1, \cdots, A_n$ and any unitarily invariant norm $|||\cdot|||$ on $\mathbb{B}(\mathscr{H})$

$$\left|\left|\left|\sum_{j=1}^{n} f(A_j)\right|\right|\right| \geq \left|\left|\left|f\left(\sum_{j=1}^{n} A_j\right)\right|\right|\right| .$$

An extension of a result of [7] for n-tuples may be stated as follows.

• If $f : [0, \infty) \to [0, \infty)$ is a convex function, then for any positive operators $A_1, \cdots, A_n$, any nonnegative numbers $\alpha_1, \cdots, \alpha_n$ with $\sum_{j=1}^{n} \alpha_j = 1$ and any unitarily invariant norm $|||\cdot|||$ on $\mathbb{B}(\mathscr{H})$

$$\left|\left|\left|f\left(\sum_{j=1}^{n} \alpha_j A_j\right)\right|\right|\right| \leq \left|\left|\left|\sum_{j=1}^{n} \alpha_j f(A_j)\right|\right|\right| . \tag{4.1}$$

Also the following result is deduced in [27] from [38].

• If $f : [0, \infty) \to [0, \infty)$ is a concave function, then for any positive operators $A_1, \cdots, A_n$, any nonnegative numbers $\alpha_1, \cdots, \alpha_n$ with $\sum_{j=1}^{n} \alpha_j = 1$ and any unitarily invariant norm $|||\cdot|||$ on $\mathbb{B}(\mathscr{H})$

$$\left|\left|\left|f\left(\sum_{j=1}^{n} \alpha_j A_j\right)\right|\right|\right| \geq \left|\left|\left|\sum_{j=1}^{n} \alpha_j f(A_j)\right|\right|\right| .$$

Some generalizations of these inequalities were obtained in [6, 12, 14, 17]:

- If f is convex with $f(0) \leq 0$ and $A \geq 0$ then for all Z with $Z^*Z \geq I$,

$$\mathrm{Tr} f(Z^*AZ) \geq \mathrm{Tr} Z^* f(A) Z.$$

- For any positive operator A and any non-negative concave function f defined on the positive half-line $[0, \infty)$,

$$|||f(Z^*AZ)||| \leq |||Z^* f(A) Z|||$$

  for all symmetric norms and all expansive operators Z (that is $Z^*Z \geq I$).
- For any positive operators $A_1, \ldots, A_n$ and any non-negative concave function f defined on the positive half-line $[0, \infty)$,

$$|||f(\sum_{i=1}^{n} Z_i^* A_i Z_i)||| \leq |||\sum_{i=1}^{n} Z_i^* f(A_i) Z_i|||$$

  for all symmetric norms and all expansive operators Z_i.

- If $f : [0, \infty) \to [0, \infty)$ is a monotone concave function, then
$$f(A + B) \leq Uf(A)U^* + V f(B)V^*,$$
for some unitaries U, V.

Recently, Bourin generalized these results in [15, 16] for $f(|.|)$ instead of f, when A, B are normal operators.

5. Equivalent conditions for the operator convexity

One of interesting problems concerning with operator inequalities is to find some equivalent conditions to the operator convexity. Most of these are some versions of Jensen's inequality. In this section, we recall these conditions; cf. [35].

Let $\mathscr{H}, \mathscr{K}$ be two Hilbert spaces and f be a continuous real function defined on an interval J. Ando [1] proved that the following conditions are equivalent to the operator convexity of f.

(1) $f(V^*AV) \leq V^*f(A)V$ for every self-adjoint operator $A \in \mathbb{B}(\mathscr{K})$ and every isometry V from $\mathscr{H}$ into $\mathscr{K}$.
(2) $f(\Phi(A)) \leq \Phi(f(A))$ for every self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ and every unital completely positive map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$.
(3) $f(\Phi(A)) \leq \Phi(f(A))$ for every self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ and every unital positive map $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$.

Hansen and Pedersen [25] proved the following significant result for the operator convexity.

Theorem 5.1. *Let f be a real continuous function on the half-open interval $[0, \alpha)$. Then the following conditions are equivalent:*

i) *f is operator convex and $f(0) \leq 0$.*
ii) *$f(A^*XA) \leq A^*f(X)A$ for every self-adjoint operator X with spectrum in $[0, \alpha)$ and every A with $\|A\| \leq 1$.*
iii) *$f(A^*XA+B^*Y B) \leq A^*f(X)A+B^*f(Y)B$ for every self-adjoint operators X, Y with spectra in $[0, \alpha)$ and every A, B with $A^*A + B^*B \leq I$.*
iv) *$f(PXP) \leq Pf(X)P$ for every projection P and every self-adjoint operator X with spectrum in $[0, \alpha)$.*

In addition, in [26] by using a unitary dilation (a unitary in $B(\mathscr{H}^n)$) of an n-tuple $(A_1, \ldots, A_n)$ with $\sum_{i=1}^n A_i^*A_i = I$, they added the following equivalent conditions to the Choi–Davis theorem.

(4) $f(\sum_{i=1}^n A_i^* X_i A_i) \leq \sum_{i=1}^n A_i^* f(X_i) A_i$ for every self-adjoint operators $X_1, \dots, X_n$ and every operators $A_1, \dots, A_n$ with $\sum_{i=1}^n A_i^* A_i = I$.

(5) $f(V^* X V) \leq V^* f(X) V$ for every self-adjoint operator X and every isometry $V \in \mathbb{B}(\mathscr{H})$.

(6) $Pf(PXP + s(I-P))P \leq Pf(X)P$ for every projection P and every number $s \in J$.

Finally, J.I. Fujii and M Fujii [23] proved the following new conditions.

(7) $f(\sum_{i=1}^n A_i^* X_i A_i) \leq \sum_{i=1}^n A_i^* f(X_i) A_i$ for self-adjoint operators $X_1, \dots, X_n \in \mathbb{B}(\mathscr{K})$ and operators $A_1, \dots, A_n \in \mathbb{B}(\mathscr{H}, \mathscr{K})$ with $\sum_{i=1}^n A_i^* A_i = I$.

(8) $f(\sum_{i=1}^n P_i A_i P_i) \leq \sum_{i=1}^n P_i f(A_i) P_i$ for projections $P_i \in \mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n P_i = I$.

Some of these results were generalized to the 2-variables case by Aujla [5] and then by Hansen [24] and to functions of several variables by Araki and Hansen [4].

There are still other characterizations of operator convexity. Moslehian [30] gave a characterization of operator convexity concerning with an operator version of Hua's inequality. Tikhonov [37] proved that a real-valued function f defined on an interval J of the real line is matrix convex if and only if for any natural k, for all families of positive operators $\{A_i\}_{i=1}^k$ in a finite-dimensional Hilbert space, such that $\sum_{i=1}^k A_i = I$, and arbitrary numbers $x_i \in J$, the inequality

$$f\left(\sum_{i=1}^k x_i A_i\right) \leq \sum_{i=1}^k f(x_i) A_i$$

holds.

6. Refinements of Jensen's inequality

We first slightly improve the conditions on (2.4).

Theorem 6.1. *If f is an operator convex function on an interval J, then*

$$f(\Phi(A)) \leq \Phi(f(A))$$

for every self-adjoint operator A in a C^*-algebra $\mathscr{A}$ with spectrum in J and every positive linear map $\Phi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ with $0 < \Phi(I) \leq I$.

Proof. We may assume that J contains 0 and $f(0) \leq 0$. The general case follows from this one by considering $g(t) = f(t + t_0) - f(t_0)$ and translation of J to $J - \{t_0\}$, where t_0 is an arbitrary point of J. Assume that $0 < \Phi(I) \leq I$. Then $\Psi(A) = \Phi(I)^{-1/2}\Phi(A)\Phi(I)^{-1/2}$ is a unital positive map. Hence, by (2.4),

$$f\left(\Phi(I)^{1/2}\Psi(A)\Phi(I)^{1/2}\right) \leq \Phi(I)^{1/2} f(\Psi(A))\Phi(I)^{1/2} \leq \Phi(I)^{1/2}\Psi(f(A))\Phi(I)^{1/2},$$

whence $f(\Phi(A)) \leq \Phi(f(A))$. □

Theorem 6.2. *Let $\Phi_1, \cdots, \Phi_n$ be strictly positive linear maps from a C^*-algebra $\mathscr{A}$ into a C^*-algebra $\mathscr{B}$, let $\Phi = \sum_{i=1}^n \Phi_i$ be unital. If f is an operator convex function on an interval J, then for every self-adjoint operator $A \in \mathscr{A}$ with spectrum contained in J, the following refinement of the Choi–Davis–Jensen inequality holds.*

$$f(\Phi(A)) \leq \sum_{i=1}^n \Phi_i(I)^{\frac{1}{2}} f\left(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}\right)\Phi_i(I)^{\frac{1}{2}} \leq \Phi(f(A)). \tag{6.1}$$

For the concave operator functions, the inequalities will be reversed.

Proof. We can simply write

$$f(\Phi(A)) = f\left(\sum_{i=1}^n \Phi_i(A)\right) = f\left(\sum_{i=1}^n \Phi_i(I)^{\frac{1}{2}}(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}})\Phi_i(I)^{\frac{1}{2}}\right).$$

Since $\sum_{i=1}^n \Phi_i(I) = I$, from (2.4), the first inequality follows.

For the second inequality, let $\Psi_i(A) = \Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}$. Then Ψ_i is a unital positive linear map. Again by applying the Choi–Davis–Jensen inequality, we have $f(\Psi_i(A)) \leq \Psi_i(f(A))$, so

$$f\left(\Phi_i(I)^{-\frac{1}{2}}\Phi_i(A)\Phi_i(I)^{-\frac{1}{2}}\right) \leq \Phi_i(I)^{-\frac{1}{2}}\Phi_i(f(A))\Phi_i(I)^{-\frac{1}{2}}.$$

Summing these inequalities over i from 1 to n, the second inequality will be obtained. □

Remark 6.3. Let Φ be a unital positive linear map from a C^*-algebra $\mathscr{A}$ into a C^*-algebra $\mathscr{B}$. If f is a non-negative operator concave function on $[0,\infty)$ and σ is its corresponding operator mean via the Kubo–Ando theory, then for every positive operator $A \in \mathscr{A}$, inequality (6.1) can be restated as

$$\Phi(f(A)) \leq \sum_{i=1}^n \big(\Phi_i(I)\sigma\Phi_i(A)\big) \leq f(\Phi(A)).$$

The next corollaries are of special interest.

Corollary 6.4. *Let $\mathscr{X}$ be a Hilbert C^*-module and $T_1, \cdots, T_n \in \mathcal{L}(\mathscr{X})$ be self-adjoint operators with spectra contained in J. Then*

$$f\left(\sum_{i=1}^n \langle x_i, T_i x_i\rangle\right) \leq \sum_{i=1}^n |x_i| f\left(|x_i|^{-1}\langle x_i, T_i x_i\rangle |x_i|^{-1}\right)|x_i| \leq \sum_{i=1}^n \langle x_i, f(T_i)x_i\rangle,$$

for every elements $x_1, \cdots, x_n \in \mathscr{X}$ with $|x_i| > 0$ and $\sum_{i=1}^n |x_i|^2 = I$.

Proof. For $1 \leq i \leq n$, define $\Phi_i : \oplus_{k=1}^n \mathcal{L}(\mathscr{X}) \to \mathscr{A}$ by $\Phi_i(\{T_k\}_{k=1}^n) = \langle x_i, T_i x_i \rangle$. Then Φ_i is a positive map and $\Phi_i(\{I\}_{k=1}^n) = |x_i|^2 > 0$. Also it follows from the hypothesis that $\sum \Phi_i$ is unital. So by using inequality (6.1), the desired result follows. □

Corollary 6.5. *If $A_1, \cdots, A_n$ are self-adjoint elements in a C^*-algebra $\mathscr{A}$ and $U_i \in \mathscr{A}$ such that $\sum_{i=1}^n U_i^* U_i = I$ and $U_i^* U_i > 0$, then*

$$f\left(\sum_{i=1}^n U_i^* A_i U_i\right) \leq \sum_{i=1}^n |U_i| f\left(|U_i|^{-1} U_i^* A_i U_i |U_i|^{-1}\right) |U_i| \leq \sum_{i=1}^n U_i^* f(A_i) U_i.$$

Proof. As in Corollary 6.4, for $\{A_i\}_{i=1}^n \in \oplus_{i=1}^n \mathscr{A}$, set $\Phi_i(A) = U_i^* A_i U_i$ in inequality (6.1). □

Since the functions $f(t) = t^{-1}$ and $f(t) = t^p$ for $p \in [1, 2]$ are operator convex and $f(t) = t^p$ for $p \in [0, 1]$ is an operator concave function, the following inequalities holds.

Corollary 6.6. *If $\mathscr{X}$ is a Hilbert C^*-module and $T_1, \cdots, T_n$ are positive operators in $\mathcal{L}(\mathscr{X})$ and $x_1, \cdots, x_n \in \mathscr{X}$ with $|x_i| > 0$ and $\sum_{i=1}^n |x_i|^2 = I$, then*

(1) $(\sum_{i=1}^n \langle x_i, T_i x_i \rangle)^{-1} \leq \sum_{i=1}^n |x_i|^2 (\langle x_i, T_i x_i \rangle)^{-1} |x_i|^2 \leq \sum_{i=1}^n \langle x_i, T_i^{-1} x_i \rangle \qquad (T_i > 0)$;
(2) $(\sum_{i=1}^n \langle x_i, T_i x_i \rangle)^p \leq \sum_{i=1}^n |x_i| (|x_i|^{-1} \langle x_i, T_i x_i \rangle |x_i|^{-1})^p |x_i| \leq \sum_{i=1}^n \langle x_i, T_i^p x_i \rangle \quad p \in [1, 2]$,
(3) $(\sum_{i=1}^n \langle x_i, T_i x_i \rangle)^p \geq \sum_{i=1}^n |x_i| (|x_i|^{-1} \langle x_i, T_i x_i \rangle |x_i|^{-1})^p |x_i| \geq \sum_{i=1}^n \langle x_i, T_i^p x_i \rangle \quad p \in [0, 1]$.

Corollary 6.7. *For positive elements $A_1, \cdots, A_n$ in a C^*-algebra $\mathscr{A}$ and each $U_i \in \mathscr{A}$ such that $\sum_{i=1}^n U_i^* U_i = I$ and $U_i^* U_i > 0$,*

(1') $(\sum_{i=1}^n U_i^* A_i U_i)^{-1} \leq \sum_{i=1}^n |U_i|^2 (U_i^* A_i U_i)^{-1} |U_i|^2 \leq \sum_{i=1}^n U_i^* A_i^{-1} U_i \qquad (A_i > 0)$;
(2') $(\sum_{i=1}^n U_i^* A_i U_i)^p \leq \sum_{i=1}^n |U_i| (|U_i|^{-1} U_i^* A_i U_i |U_i|^{-1})^p |U_i| \leq \sum_{i=1}^n U_i^* A_i^p U_i \quad p \in [1, 2]$,
(3') $(\sum_{i=1}^n U_i^* A_i U_i)^p \geq \sum_{i=1}^n |U_i| (|U_i|^{-1} U_i^* A_i U_i |U_i|^{-1})^p |U_i| \geq \sum_{i=1}^n U_i^* A_i^p U_i \quad p \in [0, 1]$.

Now by an example we show that inequality (6.1) is a refinement of the Choi–Davis–Jensen inequality. More precisely, there are some examples for which both inequalities in (6.1) are strict.

Example 6.8. Let $x_1, \cdots, x_n$ be elements of a Hilbert C^*-module $\mathscr{X}$ such that $|x_i| > 0$ and $\sum_{i=1}^n |x_i|^2 = I$. For $1 \leq i \leq n$, set $T_i = y_i \otimes y_i$, where $y_1, \cdots, y_n \in \mathscr{X}$. From Corollary 6.6 (2) with $p = 2$, we have

$$\Big(\sum_{i=1}^n |\langle y_i, x_i \rangle|^2\Big)^2 \leq \sum_{i=1}^n |\langle y_i, x_i \rangle|^2 |x_i|^{-2} |\langle y_i, x_i \rangle|^2 \leq \sum_{i=1}^n |y_i \langle y_i, x_i \rangle|^2. \tag{6.2}$$

Let $\mathscr{H}$ be a Hilbert space of dimension greater than 3, e_1, e_2, e_3 be orthonormal vectors in $\mathscr{H}$ and $x_i = \frac{\sqrt{2}}{2} e_i$ and $y_i = i e_i + e_3$, for $i = 1, 2$. A straightforward computation shows that

$\big(\sum_{i=1}^n |\langle y_i, x_i\rangle|^2\big)^2 = \frac{25}{4}, \sum_{i=1}^n |\langle y_i, x_i\rangle|^2 |x_i|^{-2} |\langle y_i, x_i\rangle|^2 = \frac{17}{2}$ and $\sum_{i=1}^n |y_i \langle y_i, x_i\rangle|^2 = 11$. Thus

$$\Big(\sum_{i=1}^n |\langle y_i, x_i\rangle|^2\Big)^2 < \sum_{i=1}^n |\langle y_i, x_i\rangle|^2 |x_i|^{-2} |\langle y_i, x_i\rangle|^2 < \sum_{i=1}^n |y_i \langle y_i, x_i\rangle|^2.$$

As another application of inequality (6.1), we have the following refinement of Choi's inequality [21, Proposition 4.3].

Theorem 6.9. *Suppose that $\Phi_1, \cdots, \Phi_n$ are strictly positive linear maps from a C^*-algebra $\mathscr{A}$ into a C^*-algebra $\mathscr{B}$ and $\Phi = \sum_{i=1}^n \Phi_i$. Then for every self-adjoint element S and $T > 0$ in $\mathscr{A}$,*

$$\Phi(S)\Phi(T)^{-1}\Phi(S) \leq \sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \leq \Phi(ST^{-1}S).$$

Proof. Set $\Psi_i(X) = \Phi(T)^{-1/2}\Phi_i(T^{1/2}XT^{1/2})\Phi(T)^{-1/2}$ and $\Psi = \sum_{i=1}^n \Psi_i$. Then Ψ_i's are strictly positive linear maps and Ψ is unital. It follows from the operator convexity of $f(t) = t^2$ and Theorem 6.2 that

$$\Psi(X)^2 \leq \sum_{i=1}^n \Psi_i(X)\Psi_i(I)^{-1}\Psi_i(X) \leq \Psi(X^2),$$

for every positive element X. Now if $X = T^{-1/2}ST^{-1/2}$ we get

$$\begin{aligned}\Phi(T)^{-1/2}\Phi(S)\Phi(T)^{-1}\Phi(S)\Phi(T)^{-1/2} &\leq \textstyle\sum_{i=1}^n \Phi(T)^{-1/2}\Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S)\Phi(T)^{-1/2} \\ &\leq \Phi(T)^{-1/2}\Phi(ST^{-1}S)\Phi(T)^{-1/2},\end{aligned}$$

or

$$\Phi(S)\Phi(T)^{-1}\Phi(S) \leq \sum_{i=1}^n \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \leq \Phi(ST^{-1}S),$$

as desired. □

Remark 6.10. The second inequality is a simple result of Choi–Davis–Jensen's inequality. It is a well-known theorem that for operators R, S, T on a Hilbert space $\mathscr{H}$, if T is invertible then

$$\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \geq 0 \Leftrightarrow R \geq S^*T^{-1}S.$$

Thus for each $1 \leq i \leq n$,

$$\begin{bmatrix} \Phi_i(T) & \Phi_i(S) \\ \Phi_i(S) & \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \end{bmatrix} \geq 0.$$

If we sum these matrices over i, we obtain

$$\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S) & \sum_{i=1}^{n} \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \end{bmatrix} \geq 0,$$

or equivalently

$$\sum_{i=1}^{n} \Phi_i(S)\Phi_i(T)^{-1}\Phi_i(S) \geq \Phi(S)\Phi(T)^{-1}\Phi(S).$$

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