A CONCEPT OF SYNCHRONICITY ASSOCIATED WITH CONVEX FUNCTIONS IN LINEAR SPACES AND APPLICATIONS

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Abstract. A concept of synchronicity associated with convex functions in linear spaces and a Čebyšev type inequality are given. Applications for norms, semi-inner products and for convex functions of several real variables are also given.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \dots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \dots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

For refinements of the Jesen inequality and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalised triangle inequality, the f-divergence measures etc. see [1]-[7].

Assume that $f : X \to \mathbb{R}$ is a *convex function* on the real linear space X. Since for any vectors $x, y \in X$ the function $g_{x,y} : \mathbb{R} \to \mathbb{R}$, $g_{x,y}(t) := f(x + ty)$ is convex it follows that the following limits exist

$$\nabla_{+(-)} f(x)(y) := \lim_{t \to 0+(-)} \frac{f(x + ty) - f(x)}{t}$$

and they are called the *right(left) Gâteaux derivatives* of the function f in the point x over the direction y.

It is obvious that for any $t > 0 > s$ we have

$$\begin{aligned} \frac{f(x+ty) - f(x)}{t} &\ge \nabla_+ f(x)(y) = \inf_{t>0}\left[\frac{f(x+ty) - f(x)}{t}\right] \\ &\ge \sup_{s<0}\left[\frac{f(x+sy) - f(x)}{s}\right] = \nabla_- f(x)(y) \ge \frac{f(x+sy) - f(x)}{s} \end{aligned} \tag{1.2}$$

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for any $x, y \in X$ and, in particular,

$$\nabla_{-} f(u)(u-v) \geq f(u)-f(v) \geq \nabla_{+} f(v)(u-v) \tag{1.3}$$

for any $u, v \in X$. We call this *the gradient inequality* for the convex function f. It will be used frequently in the sequel in order to obtain various results related to Jensen's inequality.

The following properties are also of importance:

$$\nabla_{+} f(x)(-y)=-\nabla_{-} f(x)(y), \tag{1.4}$$

and

$$\nabla_{+(-)} f(x)(\alpha y)=\alpha \nabla_{+(-)} f(x)(y) \tag{1.5}$$

for any $x, y \in X$ and $\alpha \geq 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, i.e.,

$$\nabla_{+} f(x)(y+z) \leq \nabla_{+} f(x)(y)+\nabla_{+} f(x)(z) \tag{1.6}$$

and

$$\nabla_{-} f(x)(y+z) \geq \nabla_{-} f(x)(y)+\nabla_{-} f(x)(z) \tag{1.7}$$

for any $x, y, z \in X$.

Some natural examples can be provided by the use of normed spaces.

Assume that $(X,\|\cdot\|)$ is a real normed linear space. The function $f: X \rightarrow \mathbb{R}$, $f(x):=\frac{1}{2}\|x\|^{2}$ is a convex function which generates *the superior* and *the inferior semi-inner products*

$$\langle y, x\rangle_{s(i)}:=\lim_{t \rightarrow 0+(-)} \frac{\|x+t y\|^{2}-\|x\|^{2}}{t}.$$

For a comprehensive study of the properties of these mappings in the Geometry of Banach Spaces see the monograph [6].

For the convex function $f_{p}: X \rightarrow \mathbb{R}$, $f_{p}(x):=\|x\|^{p}$ with $p>1$, we have

$$\nabla_{+(-)} f_{p}(x)(y)=\begin{cases} p\|x\|^{p-2}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x=0\end{cases}$$

for any $y \in X$.

If $p=1$, then we have

$$\nabla_{+(-)} f_{1}(x)(y)=\begin{cases}\|x\|^{-1}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0 \\ \\ +(-)\|y\| & \text{if } x=0\end{cases}$$

for any $y \in X$.

This class of functions will be used to illustrate the inequalities obtained in the general case of convex functions defined on an entire linear space.

In the recent paper [9] the following refinement and reverse of the Jensen inequality in terms of the gradient have been obtained:

Theorem 1. *Let $f : X \to \mathbb{R}$ be a convex function defined on a linear space X. Then for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{P}^n$ we have the inequality*

$$\begin{aligned}(1.8)\quad \sum_{k=1}^{n} p_k \nabla_- f(x_k)(x_k) - \sum_{k=1}^{n} p_k \nabla_- f(x_k)\left(\sum_{i=1}^{n} p_i x_i\right) &\\ \geq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) &\\ \geq \sum_{k=1}^{n} p_k \nabla_+ f\left(\sum_{i=1}^{n} p_i x_i\right)(x_k) - \nabla_+ f\left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i x_i\right) &\geq 0.\end{aligned}$$

A particular case of interest is for $f(x) = \|x\|^p$ where $(X, \|\cdot\|)$ is a normed linear space. Then for any $p \geq 1$, for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{P}^n$ with $\sum_{i=1}^{n} p_i x_i \neq 0$ we have the inequality

$$\begin{aligned}(1.9)\quad \sum_{i=1}^{n} p_i \|x_i\|^p - \left\|\sum_{i=1}^{n} p_i x_i\right\|^p &\\ \geq p\left\|\sum_{i=1}^{n} p_i x_i\right\|^{p-2}\left[\sum_{k=1}^{n} p_k \left\langle x_k, \sum_{j=1}^{n} p_j x_j\right\rangle_s - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right] &\geq 0.\end{aligned}$$

If $p \geq 2$ the inequality holds for any n-tuple of vectors and probability distribution.

Also, for any $p \geq 1$, for any n-tuple of vectors $\mathbf{x} = (x_1, ..., x_n) \in X^n \setminus \{(0, ..., 0)\}$ and any probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{P}^n$ we have the inequality

$$\begin{aligned}(1.10)\quad p\left[\sum_{k=1}^{n} p_k \|x_k\|^p - \sum_{k=1}^{n} p_k \|x_k\|^{p-2}\left\langle \sum_{i=1}^{n} p_i x_i, x_k\right\rangle_i\right] &\\ \geq \sum_{i=1}^{n} p_i \|x_i\|^p - \left\|\sum_{i=1}^{n} p_i x_i\right\|^p &.\end{aligned}$$

Motivated by the above results we introduce in this paper a class of sequences associated with convex functions in linear spaces and establish a Čebyšev type inequality and some new inequalities for convex functions. Applications for norms, semi-inner products and for convex functions of several real variables are also given.

2. $\nabla f-$Synchronicity

Consider $f : X \to \mathbb{R}$ a convex function on the linear space X. We also assume that $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are two $n-$tuples of vectors with $u_i, v_i \in X$, $i \in \{1, \ldots, n\}$.

Definition 1. *We say that v is $\nabla f-$synchronous with u if*

$$(2.1)\qquad \nabla_- f(u_k)(v_k - v_j) \geq \nabla_+ f(u_j)(v_k - v_j)$$

for any $k, j \in \{1, \ldots, n\}$. If the inequality is reversed in (2.1) for each $k, j \in \{1, \ldots, n\}$, then we say that v is $\nabla f-$asynchronous with u.

We notice that in general, if v is $\nabla f-$asynchronous with u, this does not imply that u is $\nabla f-$synchronous with v.

As general examples of such convex functions we can consider $f(x) = \|x\|^p$, $p \geq 1$ where $(X, \|\cdot\|)$ is a normed linear space. Since (see Introduction)

$$\nabla_- f(x)(y) = p\|x\|^{p-2}\langle y, x\rangle_i \quad \text{for } x, y \in X \quad \text{with } x \neq 0;$$

$$\nabla_- f(0)(y) = \begin{cases} 0 & \text{if } p > 1 \\ \\ -\|y\| & \text{if } p = 1 \end{cases}, \quad \text{for } y \in X;$$

$$\nabla_+ f(x)(y) = p\|x\|^{p-2}\langle y, x\rangle_s \quad \text{for } x, y \in X \quad \text{with } x \neq 0;$$

$$\nabla_+ f(0)(y) = \begin{cases} 0 & \text{if } p > 1 \\ \\ \|y\| & \text{if } p = 1 \end{cases}, \quad \text{for } y \in X,$$

where $\langle\cdot,\cdot\rangle_s$ is the *superior semi-inner product* and $\langle\cdot,\cdot\rangle_i$ is the *inferior semi-inner product*, then we can define the following concepts of synchronicity for the two $n-$tuples of vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

Let $p \geq 1$ and $u, v \in X^n$ be as above. We say that v *is* $p-\nabla-$*synchronous with* u if

$$\|u_k\|^{p-2}\langle v_k - v_j, u_k\rangle_i \geq \|u_j\|^{p-2}\langle v_k - v_j, u_j\rangle_s \tag{2.2}$$

for any $k, j \in \{1, \ldots, n\}$.

We observe that for $p \in [1, 2)$ we should assume that $u_k \neq 0$ for $k \in \{1, \ldots, n\}$. For $p = 2$, the equation (2.2) reduces to

$$\langle v_k - v_j, u_k\rangle_i \geq \langle v_k - v_j, u_j\rangle_s \quad \text{for any } k, j \in \{1, \ldots, n\}. \tag{2.3}$$

If $(X, \|\cdot\|)$ is a smooth normed space, meaning that the norm is Gâteaux differentiable on any $x \in X$, $x \neq 0$ and if we denote by $[\cdot,\cdot]$ the semi-inner product generating the norm $\|\cdot\|$ (see [6, pp. 19-20]), then the fact that v is $p-\nabla-$synchronous with u means that

$$\|u_k\|^{p-2}[v_k - v_j, u_k] \geq \|u_j\|^{p-2}[v_k - v_j, u_j] \tag{2.4}$$

for any $k, j \in \{1, \ldots, n\}$. For $p = 2$, we have

$$[v_k - v_j, u_k] \geq [v_k - v_j, u_j] \quad \text{for any } k, j \in \{1, \ldots, n\}. \tag{2.5}$$

Moreover, if the norm $\|\cdot\|$ is generated by an inner product $\langle\cdot,\cdot\rangle$, then v is $p-\nabla-$synchronous with u means that

$$\left\langle v_k - v_j, \|u_k\|^{p-2} u_k - \|u_j\|^{p-2} v_j \right\rangle \geq 0 \quad \text{for any } k, j \in \{1, \ldots, n\} \tag{2.6}$$

while for $p = 2$, it reduces to

$$\langle v_k - v_j, u_k - u_j\rangle \geq 0 \quad \text{for any } k, j \in \{1, \ldots, n\}, \tag{2.7}$$

which is the concept of *synchronous sequences* in inner product spaces that has been introduced in [13]. For some inequalities for synchronous sequences in inner product spaces, see [13] and [14].

As some natural examples of synchronous sequences in inner product spaces, we can consider the sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{Ax_i\}_{i\in\mathbb{N}}$ where $A : X \to X$ is a positive linear operator on X, i.e., $\langle Ax, x\rangle \geq 0$ for any $x \in X$.

For a convex function $f : X \to \mathbb{R}$ we define $\tilde{\nabla} f(\cdot)(\cdot)$ as

$$\tilde{\nabla} f(x)(y) := \frac{1}{2}\left[\nabla_{-} f(x)(y) + \nabla_{+} f(x)(y)\right], \tag{2.8}$$

where $x, y \in X$.

We observe that for f as above, we have *the homogeneity property*:

$$\tilde{\nabla} f(x)(\alpha y) = \alpha \tilde{\nabla} f(x)(y) \quad \text{for any } x, y \in X, \tag{2.9}$$

and any $\alpha \in \mathbb{R}$.

The following inequality for $\nabla - f-$synchronous sequences holds.

Theorem 2. *Assume that v is $\nabla - f-$synchronous with u and $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability distribution. Then*

$$\sum_{i=1}^{n} p_i \tilde{\nabla} f(u_i)(v_i) \geq \sum_{i,j=1}^{n} p_i p_j \tilde{\nabla} f(u_i)(v_j). \tag{2.10}$$

Proof. Since $\nabla_{+}(\cdot)(\cdot)$ is subadditive in the second variable, then we have

$$\nabla_{+} f(u_i)(v_i - v_j) \geq \nabla_{+} f(u_i)(v_i) - \nabla_{+} f(u_i)(v_j) \tag{2.11}$$

for any $i, j \in \{1, \ldots, n\}$.

Also, by the fact that $\nabla_{-}(\cdot)(\cdot)$ is superadditive in the second variable, we have that

$$\nabla_{-} f(u_i)(v_i) - \nabla_{-} f(u_i)(v_j) \geq \nabla_{-} f(u_i)(v_i - v_j) \tag{2.12}$$

for all $i, j \in \{1, \ldots, n\}$.

Now, by (2.11), (2.12) and by the definition of $\nabla - f-$synchronicity, we deduce that

$$\nabla_{-} f(u_i)(v_i) - \nabla_{-} f(u_i)(v_j) \geq \nabla_{+} f(u_i)(v_i) - \nabla_{+} f(u_i)(v_j),$$

which is equivalent with

$$\nabla_{-} f(u_i)(v_i) + \nabla_{+} f(u_i)(v_j) \geq \nabla_{+} f(u_i)(v_i) + \nabla_{-} f(u_i)(v_j) \tag{2.13}$$

for all $i, j \in \{1, \ldots, n\}$.

Therefore, by multiplying (2.13) with $p_i p_j \geq 0$ and summing over i and j from 1 to n, we get

$$\begin{aligned} \sum_{i=1}^{n} p_i \nabla_{-} f(u_i)(v_i) &+ \sum_{j=1}^{n} p_j \nabla_{+} f(u_i)(v_j) \\ &\geq \sum_{i,j=1}^{n} p_i p_j \nabla_{+} f(u_i)(v_i) + \sum_{i,j=1}^{n} p_i p_j \nabla_{-} f(u_i)(v_j). \end{aligned} \tag{2.14}$$

Now, observe that

$$\sum_{j=1}^{n} p_j \nabla_{+} f(u_j)(v_j) = \sum_{i=1}^{n} p_i \nabla_{+} f(u_i)(v_i)$$

and

$$\sum_{i,j=1}^{n} p_i p_j \nabla_{+} f(u_j)(v_i) = \sum_{i,j=1}^{n} p_i p_j \nabla_{+} f(u_i)(v_j),$$

which, by (2.14) divided by 2, provides the desired result (2.10). □

Corollary 1. *With the assumptions of Theorem 2 and, if in addition* $\tilde{\nabla} f(u_i)(\cdot)$ *is additive for any* $i \in \{1, \ldots, n\}$, *then we have*

$$\sum_{i=1}^{n} p_i \tilde{\nabla} f(u_i)(v_i) \geq \sum_{i,j=1}^{n} p_i p_j \tilde{\nabla} f(u_i)\left(\sum_{j=1}^{n} p_j u_j\right). \tag{2.15}$$

Remark 1. *If* f *is Gâteaux differentiable at the points* u_i, $i \in \{1, \ldots, n\}$, *then* $\tilde{\nabla} f(u_i)(\cdot) = \nabla f(u_i)(\cdot)$ *and is therefore linear on* X. *With this assumption, the inequality (2.15) holds with* ∇ *instead of* $\tilde{\nabla}$. *Moreover, there are examples of convex functions defined on linear spaces for which* $\tilde{\nabla} f(x)(\cdot)$ *is linear for any* $x \neq 0$ *without the function* f *being Gâteaux differentiable at that point (see* [6, pp. 44-45]*).*

Following [15], we consider the $g-$*semi-inner product* $\langle \cdot, \cdot \rangle_g : X \times X \to \mathbb{R}$ defined by

$$\langle y, x \rangle_g := \frac{1}{2}\left[\langle y, x \rangle_i + \langle y, x \rangle_s\right], \quad x, y \in X.$$

Utilising this notation, we have the following conditional inequality for semi-inner products and norms in normed linear spaces.

Proposition 1. *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X^n$ *and* $p \geq 1$. *If*

$$\|u_k\|^{p-2} \langle v_k - v_j, u_k \rangle_i \geq \|u_j\|^{p-2} \langle v_k - v_j, u_j \rangle_s \tag{2.16}$$

for any $k, j \in \{1, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} p_k \|u_k\|^{p-2} \langle v_k, u_k \rangle_g \geq \sum_{k,j=1}^{n} p_k p_j \|u_k\|^{p-2} \langle v_j, u_k \rangle_g \tag{2.17}$$

for any $\mathbf{p}$ *a probability distribution. If* $p \geq 2$, *then we should have in (2.16) all* $u_k \neq 0$. *If* $p = 2$ *and*

$$\langle v_k - v_j, u_k \rangle_i \geq \langle v_k - v_j, u_j \rangle_s \tag{2.18}$$

for any $k, j \in \{1, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} p_k \langle v_k, u_k \rangle_g \geq \sum_{k,j=1}^{n} p_k p_j \langle v_j, u_k \rangle_g, \tag{2.19}$$

for any $\mathbf{p}$ *a probability distribution.*

As a particular case of interest, we state the following result that holds in inner product spaces.

Corollary 2. *Let* $(X, \langle \cdot, \cdot \rangle)$ *be a real inner product space,* $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X^n$ *and* $p \geq 1$. *If*

$$\left\langle v_k - v_j, \|u_k\|^{p-2} u_k - \|u_j\|^{p-2} v_j \right\rangle \geq 0 \tag{2.20}$$

for any $k, j \in \{1, \ldots, n\}$, *then*

$$\sum_{k=1}^{n} p_k \|u_k\|^{p-2} \langle v_k, u_k \rangle \geq \left\langle \sum_{j=1}^{n} p_j u_j, \sum_{k=1}^{n} p_k \|u_k\|^{p-2} u_k \right\rangle \tag{2.21}$$

for any $\mathbf{p}$ *a probability distribution.*

Remark 2. *We observe that if the $n-$tuples u and v above are synchronous, i.e.,*

$$\langle v_k - v_j, u_k - u_j \rangle \geq 0 \quad \text{for any } j, k \in \{1, \ldots, n\}, \tag{2.22}$$

then we have the following Čebyšev type inequality

$$\sum_{k=1}^{n} p_k \langle v_k, u_k \rangle \geq \left\langle \sum_{k=1}^{n} p_k v_k, \sum_{k=1}^{n} p_k u_k \right\rangle. \tag{2.23}$$

This result was first obtained in [13].

3. Inequalities for Convex Functions

The following result for convex functions may be stated:

Theorem 3. *Let $f : X \to \mathbb{R}$ be a convex function on the linear space X and $x, y \in X^n$. Let $\mathbf{p}$ be a probability distribution so that $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$. If $x-y$ is $\tilde{\nabla} - f-$synchronous with y and $\tilde{\nabla} f(y_i)(\cdot)$ is additive for each $i \in \{1, \ldots, n\}$, then we have the inequality:*

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i). \tag{3.1}$$

Proof. Since f is convex, then for any $x, y \in X$ we have

$$f(x) - f(y) \geq \nabla_+ f(y)(x-y) \geq \tilde{\nabla} f(y)(x-y). \tag{3.2}$$

Then from (3.2) we have the inequality:

$$f(x_i) - f(y_i) \geq \tilde{\nabla} f(y_i)(x_i - y_i) \tag{3.3}$$

for each $i \in \{1, \ldots, n\}$.

Now, if we multiply (3.3) with $p_i \geq 0$ and then sum over i from 1 to n, we get

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geq \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(x_i - y_i). \tag{3.4}$$

Now, if we use Corollary 1 for $u_i = y_i$ and $v_i = x_i - y_i$, $i \in \{1, \ldots, n\}$, we deduce the inequality

$$\begin{aligned} \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(x_i - y_i) &\geq \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)\left(\sum_{i=1}^{n} p_i (x_i - y_i)\right) \\ &= \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(0) = 0. \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5), we deduce the desired inequality (3.1). □

Remark 3. *It is clear that if f is Gâteaux differentiable at all the points y_i, $i \in \{1, \ldots, n\}$, then $\tilde{\nabla} f(y_i)(\cdot) = \nabla f(y_i)(\cdot)$, $i \in \{1, \ldots, n\}$, which are linear on X.*

In the case of Gâteaux differentiable functions, we can state the following result as well.

Theorem 4. *Let $f : X \to \mathbb{R}$ be a convex and Gâteaux differentiable function on the linear space X. Assume that $x, y \in X^n$ and $\mathbf{p}$ is a probability distribution. If $x - y$ is $\tilde{\nabla} - f-$synchronous with y and*

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i \in \bigcap_{i=1}^{n} \ker\left(\nabla f\left(y_i\right)(\cdot)\right),$$

then

$$\sum_{i=1}^{n} p_i f\left(x_i\right) \geq \sum_{i=1}^{n} p_i f\left(y_i\right). \tag{3.6}$$

The proof is as in that of Theorem 3 when in (3.5) we take into account that

$$\nabla f\left(y_i\right)\left(\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i\right) = 0$$

for all $i \in \{1, \dots, n\}$ since

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i \in \bigcap_{i=1}^{n} \ker\left(\nabla f\left(y_i\right)(\cdot)\right).$$

The following result in smooth normed linear spaces may be stated.

Proposition 2. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and let $[\cdot,\cdot]$ be the semi-inner product that generates its norm $\|\cdot\|$. If $x, y \in X^n$ and $p \geq 1$ are such that*

$$\|y_k\|^{p-2}\left[x_k - y_k - x_j + y_j, y_k\right] \geq \|y_j\|^{p-2}\left[x_k - y_k - x_j + y_j, y_j\right] \tag{3.7}$$

for any $k, j \in \{1, \dots, n\}$, then for any probability distribution $\mathbf{p}$ with the property that

$$\sum_{j=1}^{n} p_j x_j = \sum_{j=1}^{n} p_j y_j \tag{3.8}$$

we have the inequality

$$\sum_{k=1}^{n} p_k \|x_k\|^p \geq \sum_{k=1}^{n} p_k \|y_k\|^p. \tag{3.9}$$

If $p \in [1, 2)$ we shall assume that $y_k \neq 0$ for $k \in \{1, \dots, n\}$.

If $p = 2$ and

$$\left[x_k - y_k - x_j + y_j, y_k\right] \geq \left[x_k - y_k - x_j + y_j, y_j\right] \tag{3.10}$$

for any $k, j \in \{1, \dots, n\}$, then for any probability distribution $\mathbf{p}$ satisfying (3.8), we have

$$\sum_{k=1}^{n} p_k \|x_k\|^2 \geq \sum_{k=1}^{n} p_k \|y_k\|^2. \tag{3.11}$$

The case of inner product spaces is incorporated in:

Corollary 3. *Let $(X; \langle\cdot,\cdot\rangle)$ be an inner product space. If $x, y \in X^n$ and $p \geq 1$ are such that*

$$\left\langle x_k - x_j, \|y_k\|^{p-2} y_k - \|y_j\|^{p-2} y_j\right\rangle \geq \left\langle y_k - y_j, \|y_k\|^{p-2} y_k - \|y_j\|^{p-2} y_j\right\rangle \tag{3.12}$$

for any $k,j \in \{1,\dots,n\}$, *then for any* $\mathbf{p}$ *satisfying (3.8), we have the inequality (3.9).*

If $p \in [1,2)$, *then we shall assume that* $y_k \neq 0$, $k \in \{1,\dots,n\}$.

If $p = 2$ *and*

$$\langle x_k - x_j, y_k - y_j \rangle \geq \|y_k - y_j\|^2 \quad \text{for any } k,j \in \{1,\dots,n\} \tag{3.13}$$

then for any $\mathbf{p}$ *satisfying (3.8), we have the inequality (3.11).*

4. Applications for Convex Functions on $\mathbb{R}^m$

Now, consider an open and convex set C in the real linear space $\mathbb{R}^m$, $m \geq 1$. For a convex and differentiable function $f : C \to \mathbb{R}$, we have

$$\nabla f(x)(y) = \left\langle \frac{\partial f(x)}{\partial x}, y \right\rangle, \quad x \in C,\ y \in \mathbb{R}^m, \tag{4.1}$$

where

$$\frac{\partial f(x)}{\partial x} = \left(\frac{\partial f(x)}{\partial x^1}, \dots, \frac{\partial f(x)}{\partial x^m} \right), \quad x = \left(x^1, \dots, x^m\right) \in C$$

and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^m$, i.e., $\langle u, v \rangle = \sum_{k=1}^m u^i \cdot v^i$, where $u = \left(u^1, \dots, u^m\right)$ and $v = \left(v^1, \dots, v^m\right) \in \mathbb{R}^m$.

Now, if $\mathbf{v} := (v_1, \dots, v_n) \in \mathbb{R}^m$ and $\mathbf{u} := (u_1, \dots, u_n) \in C^m$, then we say that $\mathbf{v}$ is $\nabla - f-$synchronous with $\mathbf{u}$ if

$$\left\langle \frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x}, v_k - v_j \right\rangle \geq 0 \quad \text{for any } k,j \in \{1,\dots,n\}. \tag{4.2}$$

The following result may be stated.

Proposition 3. *Let* $f : C \to \mathbb{R}$ *be a differentiable convex function on the open and convex set* $C \subseteq \mathbb{R}^m$. *If* $\mathbf{v} := (v_1, \dots, v_n) \in \mathbb{R}^m$ *and* $\mathbf{u} := (u_1, \dots, u_n) \in C^m$ *are such that* $\mathbf{v}$ *is* $\nabla - f-$*synchronous with* $\mathbf{u}$, *then for any probability distribution* $\mathbf{p} = (p_1, \dots, p_n)$, *we have the inequality*

$$\sum_{i=1}^n p_i \left\langle \frac{\partial f(u_i)}{\partial x}, v_i \right\rangle \geq \left\langle \sum_{i=1}^n p_i \frac{\partial f(u_i)}{\partial x}, \sum_{i=1}^n p_i v_i \right\rangle. \tag{4.3}$$

The proof is obvious by Corollary 1.

Now, if $u_k = \left(u_k^1, \dots, u_k^m\right)$, $k \in \{1,\dots,n\}$ and $v_k = \left(v_k^1, \dots, v_k^m\right)$, then

$$\left\langle \frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x}, v_k - v_j \right\rangle = \sum_{\ell=1}^m \left(\frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x} \right) \left(v_k^\ell - v_j^\ell\right). \tag{4.4}$$

Remark 4. *The above relation (4.4) shows that a sufficient condition for* $\mathbf{v}$ *to be* $\nabla - f-$*synchronous with* $\mathbf{u}$ *is that all the sequences* $\left\{ \frac{\partial f(u_k)}{\partial x_\ell} \right\}_{k=1,\dots,n}$ *and* $\left\{ v_k^\ell \right\}_{k=1,\dots,n}$ *are synchronous, where* $\ell \in \{1,\dots,m\}$, *i.e.,*

$$\left(\frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x} \right) \left(v_k^\ell - v_j^\ell\right) \geq 0 \quad \text{for any } k,j \in \{1,\dots,n\} \tag{4.5}$$

and for all $\ell \in \{1,\dots,m\}$.

The following result is an obvious consequence of Theorem 4.

Proposition 4. *Let $f : C \to \mathbb{R}$ be a differentiable convex function on the open and convex set $C \subseteq \mathbb{R}^m$. If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, \dots, y_n) \in C^m$ are such that*

$$\left\langle \frac{\partial f(y_k)}{\partial x} - \frac{\partial f(y_j)}{\partial x}, x_k - x_j \right\rangle \geq \left\langle \frac{\partial f(y_k)}{\partial x} - \frac{\partial f(y_j)}{\partial x}, y_k - y_j \right\rangle, \tag{4.6}$$

for each $k, j \in \{1, \dots, n\}$, then for any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ with

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i \tag{4.7}$$

we have the inequality

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i). \tag{4.8}$$

Remark 5. *As above, a sufficient condition for (4.6) to hold is that the sequences $\left\{\frac{\partial f(y_k)}{\partial x_\ell}\right\}_{k=1,\dots,n}$ and $\left\{x_k^\ell - y_k^\ell\right\}_{k=1,\dots,n}$ are synchronous for each $\ell \in \{1, \dots, m\}$.*

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