

THE BEST BOUNDS IN VERNESCU'S INEQUALITIES FOR THE EULER'S CONSTANT

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ABSTRACT. We present a survey of some results on the inequalities for the Euler's constant and prove the following result: Let $\gamma = 0.577215664\dots$ be the Euler's constant, and let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{2n} - \log n$, then for all integers $n \geq 1$,

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n < \frac{1}{12(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{12\gamma-6}} - 1 = 0.038859\dots \quad \text{and} \quad b = 0.$$

We obtain the monotonicity properties of functions related to the psi function. Relevant connections of the results presented here with those derived in earlier works are also pointed out.

1. INTRODUCTION

The Euler's constant $\gamma = 0.577215664\dots$ is defined as the limit of the sequence

$$D_n = \sum_{k=1}^n \frac{1}{k} - \log n. \tag{1}$$

It is also known as the Euler-Mascheroni constant. Several bounds for $D_n - \gamma$ have been given in [2, 3, 5, 18-22, 24], see also [7]. From the estimations [22]

$$\frac{1}{2n+1} < D_n - \gamma < \frac{1}{2n} \quad \text{for } n \geq 1,$$

it follows that the convergence speed of the sequence D_n ,

$$\lim_{n \rightarrow \infty} n(D_n - \gamma) = \frac{1}{2}.$$

The convergence of the sequence D_n to γ is very slow. The next step in the study of the convergence speed is to find other sequences which converge faster to γ . One method is to change the logarithmic term in (1). In 1993, D. W. DeTemple [9] studied the sequence

$$R_n = \sum_{k=1}^n \frac{1}{k} - \log \left(n + \frac{1}{2} \right),$$

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and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

Now let

$$H(n) = n^2(R_n - \gamma), \quad n \geq 1.$$

Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1, \infty)$ from $H(1) = -\gamma + 1 - \log(3/2) = 0.0173\dots$ to $1/24 = 0.0416\dots$. E. A. Karatsuba [12] proved that for all integers $n \geq 1$, $H(n) < H(n+1)$, by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that $[(n+1)/n]^2 H(n)$ is a decreasing convex function [4]. The author [8] verified that for all integers $n \geq 1$, $H(n)$ and $[(n+1/2)/n]^2 H(n)$ are both strictly increasing concave sequences, while $[(n+1)/n]^2 H(n)$ is strictly decreasing log-convex sequence. Recently, the author [7] obtained the following result: For all integers $n \geq 1$, then

$$\frac{1}{24(n+a)^2} \leq R_n - \gamma < \frac{1}{24(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24[-\gamma + 1 - \log(3/2)]}} - 1 = 0.55106\dots \quad \text{and} \quad b = \frac{1}{2}.$$

In 1997, T. Negoi [17] proved that the sequence

$$T_n = \sum_{k=1}^n \frac{1}{k} - \log \left(n + \frac{1}{2} + \frac{1}{24n} \right)$$

is strictly increasing and convergent to γ . Moreover,

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \quad (2)$$

Later, A. Vernescu [23] have found a fast convergent sequence to γ , by having the idea to replace the last term of the harmonic sum. He proved that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{2n} - \log n$$

is strictly increasing and convergent to γ . Moreover,

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}. \quad (3)$$

C. Mortici and A. Vernescu [16] wrote (3) as

$$\frac{1}{2n} - \frac{1}{12n^2} < D_n - \gamma < \frac{1}{2n} - \frac{1}{12(n+1)^2}, \quad (4)$$

and pointed out that the estimations (4) are stronger than the estimations

$$\frac{1}{2n} - \frac{1}{8n^2} < D_n - \gamma < \frac{1}{2n}$$

due to J. Franel (e.g. [13, p.523]).

In view of the inequality (3) it is natural to ask: What is the smallest number a and what is the largest number b such that the inequality

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n \leq \frac{1}{12(n+b)^2}$$

holds for all integers $n \geq 1$? The following Theorem 1 answers this question.

Theorem 1. *For all integers $n \geq 1$, then*

$$\frac{1}{12(n+a)^2} \leq \gamma - x_n < \frac{1}{12(n+b)^2} \quad (5)$$

with the best possible constants

$$a = \frac{1}{\sqrt{12\gamma - 6}} - 1 = 0.038859\dots \quad \text{and} \quad b = 0.$$

In view of the inequality (2), we pose the following conjecture.

Conjecture 1. *For all integers $n \geq 1$, then*

$$\frac{1}{12(n+\alpha)^2} \leq \gamma - T_n < \frac{1}{12(n+\beta)^2} \quad (6)$$

with the best possible constants

$$\alpha = \frac{1}{\sqrt[3]{48[1 - \gamma + \log(\frac{37}{24})]}} - 1 = 0.2738\dots \quad \text{and} \quad \beta = \frac{83}{720} = 0.1152\dots$$

Recall that a function f is said to be completely monotonic over (a, b) , where $-\infty \leq a < b \leq \infty$, if

$$(-1)^n f^{(n)}(x) \geq 0, \quad a < x < b, \quad n = 0, 1, 2, \dots \quad (7)$$

If, in addition, f is continuous at $x = a$, then it is called completely monotonic over $[a, b)$, with similar definitions for $(a, b]$ and $[a, b]$. Dubourdien [10, p.98] pointed out that if a non-constant function f is completely monotonic over (a, ∞) , then strict inequality holds in (7). See also [11] for a simpler proof of this result. Recall that a function f is said to be a Bernstein function on an interval I if $f > 0$ and f' is completely monotonic on I .

Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}, \quad (8)$$

where $\psi = \Gamma'/\Gamma$ is the psi (or digamma) function, the logarithmic derivative of the gamma function, the inequality (3) can be written as

$$\frac{1}{12(n+1)^2} < \log n - \psi(n+1) + \frac{1}{2n} < \frac{1}{12n^2}, \quad n = 1, 2, \dots \quad (9)$$

Motivated by (9), we establish the following Theorem 2.

Theorem 2. (i) *The function*

$$g(x) = x^2 \left[\log x - \psi(x+1) + \frac{1}{2x} \right]$$

is a Bernstein function on $(0, \infty)$.

(ii) *The function*

$$h(x) = (x+1)^2 \left[\log x - \psi(x+1) + \frac{1}{2x} \right] \quad (10)$$

is strictly decreasing on $(0, \infty)$.

Now we pose the following conjecture.

Conjecture 2. *The function $h(x)$ defined by (10) is completely monotonic on $(0, \infty)$.*

2. PROOFS OF THEOREMS

In order prove our Theorem 1 we need to the following results [6, pp.67-68]: Let $m \geq 0$ and $n \geq 1$ be integers, then for $x > 0$,

$$\frac{1}{2x} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j} \frac{1}{x^{2j}} < \psi(x+1) - \ln x < \frac{1}{2x} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j} \frac{1}{x^{2j}}, \quad (11)$$

$$\begin{aligned} & \frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}} \\ & < (-1)^{n+1} \psi^{(n)}(x+1) < \frac{(n-1)!}{x^n} - \frac{n!}{2x^{n+1}} + \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} \frac{\Gamma(n+2j)}{x^{n+2j}}, \end{aligned} \quad (12)$$

where B_k denotes Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} t^j.$$

The first five Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$$

In particular, taking in (11) $m = 1$ we obtain for $x > 0$,

$$\frac{1}{12x^2} - \frac{1}{120x^4} < \log x - \psi(x+1) + \frac{1}{2x} < \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6}, \quad (13)$$

and taking in (12) $m = 1$ and $n = 1$ we obtain for $x > 0$,

$$\frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2} < \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}. \quad (14)$$

Now we are in position to prove our Theorem 1.

Proof of Theorem 1. The inequality (5) can be written as

$$a \geq \frac{1}{\sqrt{12[\log n - \psi(n+1) + \frac{1}{2n}]}} - n > b.$$

In order to prove (5) we define

$$f(x) = \frac{1}{\sqrt{12[\log x - \psi(x+1) + \frac{1}{2x}]}} - x.$$

Differentiation and applying (13) and (14) yields

$$\begin{aligned}
 & \left[12 \left(\log x - \psi(x+1) + \frac{1}{2x} \right) \right]^{3/2} f'(x) \\
 &= 6 \left(\psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2} \right) - \left[12 \left(\log x - \psi(x+1) + \frac{1}{2x} \right) \right]^{3/2} \\
 &< 6 \left(\frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \right) - \left[12 \left(\frac{1}{12x^2} - \frac{1}{120x^4} \right) \right]^{3/2} \\
 &= \frac{1}{x^3} \left[\left(1 - \frac{1}{5x^2} + \frac{1}{7x^4} \right) - \left(1 - \frac{1}{10x^2} \right)^{3/2} \right].
 \end{aligned}$$

Now we show that there exists a positive real number x_0 such that the function f is strictly decreasing on (x_0, ∞) . In order to find x_0 , we consider

$$1 - \frac{1}{5x^2} + \frac{1}{7x^4} < \left(1 - \frac{1}{10x^2} \right)^{3/2}$$

By Bernoulli's inequality: Let $x \geq -1$, then for $\alpha < 0$ or $\alpha > 1$, $(1+x)^\alpha \geq 1+\alpha x$, the equal sign holds if and only if $x = 0$, we have

$$1 - \frac{3}{20x^2} < \left(1 - \frac{1}{10x^2} \right)^{3/2}.$$

The inequality

$$1 - \frac{1}{5x^2} + \frac{1}{7x^4} < 1 - \frac{3}{20x^2}$$

holds for $x > 1.69\dots$, and then, $f'(x) < 0$ for $x > 1.69\dots$. Straightforward calculation produces $f(1) = 0.038859\dots$, $f(2) = 0.0229733\dots$, thus, the sequence $f(n) = \frac{1}{\sqrt{12[\log n - \psi(n+1) + \frac{1}{2n}]}} - n$ is strictly decreasing for all integers $n \geq 1$. This leads to

$$\lim_{n \rightarrow \infty} f(n) < f(n) \leq f(1) = \frac{1}{\sqrt{12\gamma - 6}} - 1.$$

It remains to prove that

$$\lim_{n \rightarrow \infty} f(n) = 0. \tag{15}$$

From the asymptotic formula

$$\psi(x+1) = \log x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} + O(x^{-6}) \quad \text{as } x \rightarrow \infty,$$

we obtain

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{12[\log x - \psi(x+1) + \frac{1}{2x}]} - x} \\
&= \frac{1 - x\sqrt{12[\log x - \psi(x+1) + \frac{1}{2x}]}{\sqrt{12[\log x - \psi(x+1) + \frac{1}{2x}]}} \\
&= \frac{1 - x\sqrt{\frac{1}{x^2} - \frac{1}{10x^4} + O(x^{-6})}}{\sqrt{\frac{1}{x^2} - \frac{1}{10x^4} + O(x^{-6})}} \\
&= \frac{x - x\sqrt{1 - \frac{1}{10x^2} + O(x^{-4})}}{\sqrt{1 - \frac{1}{10x^2} + O(x^{-4})}} \\
&= \frac{\frac{1}{20x} + O(x^{-3})}{1 - \frac{1}{20x^2} + O(x^{-4})} \rightarrow 0 \quad \text{as } x \rightarrow \infty,
\end{aligned}$$

and then, (15) holds. The proof is complete. \square

Proof of Theorem 2. (i) Using the representations [1, p. 259]

$$\begin{aligned}
\psi(x) &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt, \\
\log x &= \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt \quad \text{and} \quad \frac{1}{x} = \int_0^\infty e^{-xt} dt,
\end{aligned}$$

we imply

$$g(x) = x^2 \int_0^\infty \omega(t) e^{-xt} dt, \quad (16)$$

where

$$\omega(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}, \quad t > 0. \quad (17)$$

Easy computations reveal that

$$\omega'(t) = \frac{(e^t - 1)^2 - t^2 e^t}{t^2 (e^t - 1)^2} > 0,$$

since

$$(e^t - 1) - te^{t/2} = \sum_{n=3}^\infty \left(1 - \frac{n}{2^{n-1}} \right) \frac{t^n}{n!} > 0.$$

Hence, for $t > 0$,

$$\omega(t) > \lim_{t \rightarrow 0} \omega(t) = 0.$$

An integration by parts yields

$$\begin{aligned}
g(x) &= -x \int_0^\infty \omega(t) d(e^{-xt}) \\
&= x \int_0^\infty \omega'(t) e^{-xt} dt \\
&= \frac{1}{12} + \int_0^\infty \omega''(t) e^{-xt} dt,
\end{aligned}$$

and then,

$$g'(x) = \int_0^{\infty} [-t\omega''(t)]e^{-xt} dt. \quad (18)$$

Easy computations reveal that

$$t^3(e^t - 1)^3\omega''(t) = t^3(e^t + e^{2t}) - 2(e^t - 1)^3.$$

Now we show that $\omega''(t) < 0$ for $t > 0$, it is sufficient to prove that

$$t^3(e^t + e^{2t}) < 2(e^t - 1)^3, \quad t > 0,$$

i.e.,

$$\cosh u < \left(\frac{\sinh u}{u}\right)^3, \quad (19)$$

where $u = \frac{t}{2}$. It is well-known (see [15, p.270] or [14, p.300]) that the inequality (19) holds for $u \neq 0$. Hence, $\omega''(t) < 0$ for $t > 0$. So, it follows from (16) and (18) that $g(x) > 0$ and g' is completely monotonic on $(0, \infty)$.

(ii) Clearly,

$$h(x) = (x+1)^2 \int_0^{\infty} \omega(t)e^{-xt} dt,$$

where ω as in (17).

It is easy to see that

$$\begin{aligned} \frac{h'(x)}{x+1} &= 2 \int_0^{\infty} \omega(t)e^{-xt} dt - x \int_0^{\infty} t\omega(t)e^{-xt} dt - \int_0^{\infty} t\omega(t)e^{-xt} dt \\ &= 2 \int_0^{\infty} \omega(t)e^{-xt} dt - \int_0^{\infty} [\omega(t) + t\omega'(t)]e^{-xt} dt - \int_0^{\infty} t\omega(t)e^{-xt} dt \\ &= \int_0^{\infty} \delta(t)e^{-xt} dt, \end{aligned}$$

with

$$\begin{aligned} \delta(t) &= (1-t)\omega(t) - t\omega'(t) \\ &= \frac{2t(e^t - 1 + t) - (t^2 - 3t + 4)(e^t - 1)^2}{2t(e^t - 1)^2} \\ &= - \sum_{n=5}^{\infty} [(n^2 - 7n + 16)2^{n-2} - 2(n^2 - 3n + 4)] \frac{t^n}{n!} < 0, \quad t > 0. \end{aligned}$$

Hence, $h'(x) < 0$ for $x > 0$. □

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