SCHUR-CONVEXITY AND SCHUR-GEOMETRIC CONVEXITY OF ČEBYŠEV FUNCTIONAL

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Abstract. The Schur-convexity or concavity and the Schur-geometric convexity or concavity of the Čebyšev Functional on the upper and lower limits of the integral are discussed. Two refinements of the Čebyšev inequality are obtained.

1. Introduction

Throughout the paper we denote the set of the real numbers and the positive real numbers by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Čebyšev functional, by [**?**, p. 43] as

$$T(f, g) := M(fg) - M(f)M(g), \tag{1}$$

where the integral mean is given by

$$M(f) := \frac{1}{b-a} \int_a^b f(x)dx. \tag{2}$$

The integrals in (1) are assumed to exist.

There are two important conclusions for the Čebyšev functional $T(f, g)$.

Theorem A. *Let $f, g : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions. If f and g are monotone in the same sense, we have the well-known Čebyšev inequality*

$$T(f, g) \geq 0. \tag{3}$$

The sign of inequality in (3) is reversed if f and g are monotone in the opposite sense.

Theorem B. *Let $f, g : [a, b] \to \mathbb{R}$ be two Lebesgue integrable functions and $m_1 \leq f \leq M_1$,$m_2 \leq g \leq M_2$. Then we have the well-known Grüss inequality*

$$|T(f, g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2). \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 26D15, 26B25.

Key words and phrases. Čebyšev Functional, Schur-convexity, Schur-geometric concavityČebyšev integral inequality.

The authors are supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM200611417009).

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1, 2, 3, 4, 9, 10] and the references cited therein.

In this paper, $T(f,g)$ is regarded as a binary function on the upper and lower limits of the integral $T:[a,b]\times[a,b]\to\mathbb{R}$,

$$\begin{aligned} T(x,y) &= T(f,g;x,y) \\ &= \frac{1}{y-x}\int_x^y f(t)g(t)dt - \left(\frac{1}{y-x}\int_x^y f(t)dt\right)\left(\frac{1}{y-x}\int_x^y g(t)dt\right). \end{aligned} \tag{5}$$

We study the Schur-convexity or concavity of $T(x,y)$ with variables (x,y) in $[a,b]\times[a,b]\subseteq\mathbb{R}^2$ and the Schur-geometrically convexity or concavity of $T(x,y)$ with variables (x,y) in $[a,b]\times[a,b]\subseteq\mathbb{R}^2_+$. We obtain the following results.

Theorem 1. *Let $f,g:[a,b]\to\mathbb{R}$ be two Lebesgue integrable functions.*

- (*i*) *If f and g are monotone in the same sense and convex or concave in the same sense. Then $T(x,y)$ is Schur-convex in $[a,b]\times[a,b]\subseteq\mathbb{R}^2$ and $T(x,y)$ is Schur-geometrically convex in $[a,b]\times[a,b]\subseteq\mathbb{R}^2_+$.*
- (*ii*) *if f and g are monotone in the opposite sense and convex or concave in the opposite sense. Then $T(x,y)$ is Schur-concave in $[a,b]\times[a,b]\subseteq\mathbb{R}^2$, and $T(x,y)$ is Schur-geometrically concave in $[a,b]\times[a,b]\subseteq\mathbb{R}^2_+$.*

Theorem 2. *Let $f,g:[a,b]\to\mathbb{R}$ be two Lebesgue integrable functions $(0<a<b)$, and let $1/2\le t_2\le t_1\le 1$ or $0\le t_1\le t_2\le 1/2$. If f and g are monotone in the same sense and convex or concave in the same sense, then*

$$\begin{aligned} 0 &\le T\left(t_2b+(1-t_2)a, t_2a+(1-t_2)b\right) \\ &\le T\left(t_1b+(1-t_1)a, t_1a+(1-t_1)b\right) \le T(a,b) \end{aligned} \tag{6}$$

if f and g are monotone in the opposite sense and convex or concave in the opposite sense, then inequalities in (5) *are all reversed.*

Theorem 3. *Let $f,g:[a,b]\to\mathbb{R}$ be two Lebesgue integrable functions $(0<a<b)$, and let $1/2\le t_2\le t_1\le 1$ or $0\le t_1\le t_2\le 1/2$. If f and g are monotone in the same sense and convex or concave in the same sense, then*

$$0\le T\left(b^{t_2}a^{1-t_2}, a^{t_2}b^{1-t_2}\right)\le T\left(b^{t_1}a^{1-t_1}, a^{t_1}b^{1-t_1}\right)\le T(a,b) \tag{7}$$

if f and g are monotone in the opposite sense and convex or concave in the opposite sense, then inequalities in (6) *are all reversed.*

Remark 1. (6) and (7) are refinements of the Čebyšev inequality (3).

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 1 ([?, ?]). Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

- (*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
- (*ii*) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for every $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(*iii*) let $\Omega \subset \mathbb{R}^n$. The function φ: $\Omega \to \mathbb{R}$ be said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2 ([?])**.** Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}^n_{++}$.The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

Definition 3 ([?, ?])**.** (*i*) $\Omega \subset \mathbb{R}^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 1 ([?, ?])**.** *Let f be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ of real numbers and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{8}$$

is known as the Hadamard's inequality for the convex function.

Lemma 2 ([?, ?])**.** *Let $\Omega \subset \mathbb{R}^n$ be a symmetric set and with a nonempty interior Ω^0, $\varphi : \Omega \to \mathbb{R}$ be a continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{9}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 3 ([?, p. 108])**.** *Let $\Omega \subset \mathbb{R}^n_+$ be symmetric with a nonempty interior geometrically convex set. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{10}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 4 ([?])**.** *Let $a \le b, u(t) = ta + (1-t)b, v(t) = tb + (1-t)a$. If $1/2 \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1/2$, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{11}$$

3. Proofs of main results

Proof of Theorem 1:

It is easy to see that $T(x, y)$ is symmetric with x, y. Without loss of generality, we may assume $x \le y$.

$$\begin{aligned}\frac{\partial T}{\partial x} =& \frac{1}{(y-x)^2}\int_x^y f(t)g(t)dt - \frac{1}{y-x}f(x)g(x)\\ &-\left(\frac{1}{(y-x)^2}\int_x^y f(t)dt - \frac{1}{y-x}f(x)\right)\left(\frac{1}{y-x}\int_x^y g(t)dt\right)\\ &-\left(\frac{1}{(y-x)^2}\int_x^y g(t)dt - \frac{1}{y-x}g(x)\right)\left(\frac{1}{y-x}\int_x^y f(t)dt\right)\\ &= \frac{1}{y-x}\left[\frac{1}{y-x}\int_x^y f(t)g(t)dt - \frac{1}{(y-x)^2}\left(\int_x^y f(t)dt\right)\left(\int_x^y g(t)dt\right)\right.\\ &\left.-\left(\int_x^y f(t)dt - f(x)\right)\left(\int_x^y g(t)dt - g(x)\right)\right]\end{aligned}$$

i.e.

$$\frac{\partial T}{\partial x} = \frac{1}{y-x}\left[T(x,y) - \left(\int_x^y f(t)dt - f(x)\right)\left(\int_x^y g(t)dt - g(x)\right)\right] \tag{12}$$

Analogous computing yields

$$\frac{\partial T}{\partial y} = \frac{1}{y-x}\left[\left(f(y) - \int_x^y f(t)dt\right)\left(g(y) - \int_x^y g(t)dt\right) - T(x,y)\right] \tag{13}$$

(i) f and g are monotone in the same sense and convex or concave in the same sense.

By Theorem A, it follows $T(x,y) \geq 0$. There are three possible cases to be determined.

Case 1. f and g are both increasing and convex in $[a,b]$.

By integral mean value theorem, there is $\xi \in (x,y)$ such that $\frac{1}{x-y}\int_x^y f(t)dt = f(\xi)$. Since f is increasing, we have

$$\frac{1}{x-y}\int_x^y f(t)dt - f(x) = f(\xi) - f(x) \geq 0.$$

By the same arguments,

$$\frac{1}{x-y}\int_x^y g(t)dt - g(x) \geq 0.$$

And then by Lemma 1, it follows that

$$\begin{aligned}&\left(\int_x^y f(t)dt - f(x)\right)\left(\int_x^y g(t)dt - g(x)\right)\\ &\leq \left(\frac{f(x)+f(y)}{2} - f(x)\right)\left(\frac{g(x)+g(y)}{2} - g(x)\right)\\ &= \frac{1}{4}(f(y)-f(x))(g(y)-g(x)).\end{aligned} \tag{14}$$

Furthermore, since f and g are both increasing in $[x,y]$, for $t \in [x,y]$ we have $f(x) \leq f(t) \leq f(y)$ and $g(x) \leq g(t) \leq g(y)$. So by Theorem B it follows that

$$T(x,y) = |T(f,g)| \leq \frac{1}{4}\left(f(y)-f(x)\right)\left(g(y)-g(x)\right). \tag{15}$$

Combining (12) with (14) and (15) yields $\frac{\partial T}{\partial x} \leq 0$. Similarly, combining (13) with (14) and (15) yields $\frac{\partial T}{\partial y} \geq 0$, and then $(x-y)\left(\frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}\right) \geq 0$. Notice that from $x \leq y$, we have $\ln x - \ln y \leq 0$, and then $(\ln x - \ln y)\left(x\frac{\partial T}{\partial x} - y\frac{\partial T}{\partial y}\right) \geq 0$. According to Lemma 2 and Lemma 3, it follows that $T(x,y)$ is Schur-convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$ and $T(x,y)$ is Schur-geometrically convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$.

Case 2. f and g are both decreasing and convex in $[a,b]$.

It is easy to see that both (14) and (15) still hold in this case, hence the conclusion same as in case 1.

Case 3. f and g are both decreasing (increasing) and concave in $[a,b]$.

Since $-f$ and $-g$ are both increasing (decreasing) and convex in $[a,b]$, from case 1 (case 2), it follows that $T(-f,-g;x,y)$ is Schur-convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$ and $T(-f,-g;x,y)$ is Schur-geometrically convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$, but $T(-f,-g;x,y) = T(f,g;x,y)$, so is $T(f,g;x,y)$.

(ii) f and g are monotone in the opposite sense and convex or concave in the opposite sense.

Since $T(x,y)$ is symmetric with f,g. Without loss of generality, we may assume f is increasing (decreasing) and convex, and g is decreasing (increasing) and concave in $[a,b]$.

Since f and $-g$ are both increasing (decreasing) and convex in $[a,b]$, from case 1 (case 2) in (i), it follows that $T(f,-g;x,y)$ is Schur-convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$ and $T(f,-g;x,y)$ is Schur-geometrically convex in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$, but $T(f,g;x,y) = -T(f,-g;x,y)$, so is $T(f,g;x,y)$ Schur-concave in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$ and $T(f,g;x,y)$ is Schur-geometrically concave in $[a,b]\times[a,b] \subseteq \mathbb{R}^2_+$.

The proof is complete. □

Proof of Theorem 2:

Combining Lemma 4 with Theorem 1,the Theorem 2 is proved. □

Proof of Theorem 3:

From Lemma 4, we have

$$\left(\ln\sqrt{ab}, \ln\sqrt{ab}\right) \prec \left(\ln(b^{t_2}a^{1-t_2}), \ln(a^{t_2}b^{1-t_2})\right) \prec \left(\ln(b^{t_1}a^{1-t_1}), \ln(a^{t_1}b^{1-t_1})\right) \prec (\ln a, \ln b). \tag{16}$$

By Theorem 1, from (16) it follows that the Theorem 3 is proved. □

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