A Note Concerning the Euler Totient

József Sándor

Abstract

We correct and improve certain results from paper [4] related to some properties of the Euler totient and similar arithmetic functions.

2000 Mathematics Subject Classification: 11A25.
Keywords: totatives of a number, Euler's totient, arithmetic functions.

1 Introduction

Let $n \geq 1$ be a positive integer, and denote by $\varphi(n)$ the number of "totatives" of n, i.e. those positive integers $r < n$ such that $r < n$. The totatives of a number (a notion due to J.J. Sylvester, from 1879) have a long history. For example, in 1888 H.W. Lloyd Tanner (see [3], [7]) studied the group G of totatives of a number, finding all its subgroups and the simple groups whose direct product is G.

It is easy to see that, the sum of totatives of n is $\dfrac{n\varphi(n)}{2}$, due to A.L. Crelle from 1845 (see [3], [7]). We note that this appears as Proposition 2 in [4]. Let $t(n) = \{k : \ 1 \leq k < n, \ (k, n) = 1\}$ be the set of totatives of n.

In 1850 A. Thacker introduced the function (see [3], [7])

$$\phi_j(n) = \sum_{t \in t(n)} t^j, \quad j \geq 0 \tag{1}$$

(i.e., the sum of jth powers of totatives of n), and noted that for $j = 0$ it reduces to Euler's totient. Clearly $\phi_1(n) = \dfrac{n\varphi(n)}{2}$.

In 1852 W. Brennecke proved that

$$\phi_2(n) = \frac{1}{3}\varphi(n) \left[n^2 + (-1)^{\omega(n)} \cdot \frac{\gamma(n)}{2} \right], \tag{2}$$

where $\omega(n)$ denotes the number of distinct prime factors of n and $\gamma(n)$ is the product of distinct prime divisors of n. Therefore, the order of $\phi_2(n)$ is essentially $\dfrac{n^2\varphi(n)}{3}$; contradicting Proposition 9 of [4]. In fact, in Proposition 9 of [4] we should have (in place of $n^2\varphi(n)/2$ on the right-hand side)

$$\phi_2(n) \geq \frac{n^2\varphi(n)}{4} \tag{3}$$

but generally speaking, (2) is much stronger than (3).

For a proof of (2), based on a simple argument of Hurwitz, see the author's paper (in cooperation) [2].

In 1851 J. Binet proved that (see [3], [7])

$$\phi_k(n) \equiv 0 \pmod{n} \text{ if } k \text{ is odd and } n \geq 3, \tag{4}$$

which for $k = 1$ gives Proposition 3 of [4].

For many other congruences and identities related to Thacker's function, see [2], [3], [7].

2 Some improvements

Proposition 11 of [4] states that

$$n\varphi(n) + 2\sigma(n) \le n^2 + n + 2, \quad n \ge 1, \tag{5}$$

where $\sigma(n)$ denotes the sum of divisors of n.

As for $n =$ prime one has $\varphi(n) = n - 1$ and $\sigma(n) = n + 1$, in (5) we get equality for $n =$ prime.

On the other hand, when n is composite, (5) can be strongly improved.

Lemma 1. *Let $d(n)$ denote the number of divisors of n. Then*

$$\frac{\sigma(n)}{d(n)} \le \frac{n+1}{2}, \quad n \ge 2 \tag{6}$$

with equality only for $n =$ prime.

One has also the inequality

$$\varphi(n) + d(n) \le n + 1, \quad n \ge 2 \tag{7}$$

with equality only for $n =$ prime or $n = 4$.

Proof. For the proof of (6), let $d_1, \dots, d_r$ be the distinct divisors of n, where $r = d(n)$. Then as $\{d_1, \dots, d_r\} = \left\{\frac{n}{d_1}, \dots, \frac{n}{d_r}\right\}$, clearly,

$$\frac{d_1 + \ldots + d_r}{r} = \frac{\left(d_1 + \frac{n}{d_1}\right) + \ldots + \left(d_r + \frac{n}{d_r}\right)}{2r}. \tag{$*$}$$

Now remark that

$$d + \frac{n}{d} \le n + 1 \tag{8}$$

for any $1 \le d \le n$, since (8) may be written also as $(d-1)(d-n) \le 0$.

Since by (8) $d_i + \frac{n}{d_i} \le n+1$, by identity $(*)$ we get at once (6). There is equality only when n has two divisors; namely $d_1 = 1$ and $d_2 = n$, when n is prime.

For the proof of (7), let us remark first that when $d > 1$ is a divisor of n, then clearly $(d, n) > 1$; so d cannot be a totative of n. Therefore, the set of divisors and the set of totatives has a single common element, namely 1.

When n is prime, then any $1 \leq k < n$ is a totative, and there are only two divisors: 1 and n, so $\varphi(n) + d(n) = n + 1$. This is true also when $n = 4$, as any number in the set

$$1 < 2 < 3 < 4$$

or is a divisor of 4, or a totative of n.

Lemma 2. *Assume that $n \neq$ prime and $n \neq 4$. Then there exists an $a \in \{1, 2, \ldots, n\}$, such that $a \nmid n$, but $(a, n) > 1$ (where $a \nmid n$ means that a doesn't divide n).*

Proof. If n is composite, and odd, let $n = N \cdot M$, where $N, M > 1$. Put $a = N \cdot m$, where $m < M$ and $(m, M) = 1$. Then clearly $a \nmid n$, but $N|a$, $N|n$, so $(n, a) > 1$.

If n is even, remark that $n - 2$ is even, too, and $n - 2 \nmid n$, if $n \neq 4$. Thus $a = n - 2$ is acceptable, as $2|a$, $2|n$.

This finishes the proof of Lemma 2.

By this lemma, the number of divisors, and the number of totients cannot be in sum greater than n, proving (7) for $n \neq$ prime and $n \neq 4$.

Remarks. 1) The inequality $\varphi(n) + d(n) \leq n$ for $n \neq$ prime, $n \neq 4$ may be satisfied with equality. For example, $n = 6$ is a solution.

2) As $n + 1 \leq \sigma(n)$, by (7) we get

$$\varphi(n) + d(n) \leq n + 1 \leq \sigma(n). \tag{10}$$

This improves the inequality $\varphi(n) + d(n) \leq \sigma(n)$ by H.D. Bagchi and M. Gupta (see [1], [8]).

Theorem 1. *If n is composite, then*

$$n\varphi(n) + 2\sigma(n) < n^2 + d(n) \tag{9}$$

Proof. By (6) we get $2\sigma(n) < (n+1)d(n)$, and by (7), $\varphi(n)+d(n) \le n$ for composite n, therefore, we can write

$$n\varphi(n)+2\sigma(n) < n\varphi(n)+(n+1)d(n) = n[\varphi(n)+d(n)]+d(n) \le n^2+d(n),$$

so (9) follows.

Remark 3. As $d(n) < 2\sqrt{n}$ (see e.g. [6], [8]), clearly (9) is an improvement of (5) for composite n.

Lemma 3. [9]

$$\sigma(n) \ge n + 1 + \sqrt{n} \cdot (d(n) - 2), \; \textit{for } n \ge 2 \tag{11}$$

with equality only for $n =$ prime or square of a prime.

Proof. When $n = p$ (prime), then $d(n) = 2$ and $\sigma(n) = n+1$; so there is equality in (10). For $n = p^2$ (square of a prime), as $\sigma(n) = p^2 + p + 1$ and $d(n) = 3$, we get again equality in (10).

Let $1 = d_1 < d_2 < \ldots < d_i < \ldots < d_k = n$ be the distinct divisors of n. As $n = \frac{n}{d_1} > \frac{n}{d_2} > \ldots > \frac{n}{d_k} = 1$ are also the divisors of n, then

$$\sigma(n) = n + 1 + \frac{1}{2}\sum_{k=1}^{n-1}\left(d_k + \frac{n}{d_k}\right) \tag{12}$$

By the arithmetic mean - geometric mean inequality one has

$$d_k + \frac{n}{d_k} \ge 2\sqrt{d_k \cdot \frac{n}{d_k}} = 2\sqrt{n},$$

with equality only if $d_n = \frac{n}{d_k}$.

Now, if $n \neq$ prime, the sum in (11) is not empty, and inequality (10) follows, as there are $d(n) - 2$ terms in the sum. There is equality only when $n = d_k^2$, and this is possibly only when $k = 2$ and d_2 is prime (since d_2 is the single divisor $d_2 < n$).

Theorem 2. *For all $n \geq 2$ one has*

$$\sigma(n) + 1 \geq n + 2 + \sqrt{n}(d(n) - 2) \geq n + d(n) \tag{13}$$

Proof. The first inequality follows by relation (10). When n is prime, as $d(n) = 2$ and $\sigma(n) = n + 1$, there is equality in each sides of (12). When $n > 1$ is composite, as $d(n) - 2 \geq 1$ and $\sqrt{n} > 1$, we have

$$n + 2 + \sqrt{n}(d(n) - 2) > n + 2 + (d(n) - 2) = n + d(n),$$

so the right side of (12) holds true with strict inequality.

Remarks. 4) Inequality $\sigma(n) + 1 \geq n + d(n)$ is Proposition 14 of [4].

5) Many other arithmetic inequalities are included also in the author's recent book [10].

References

[1] Bagchi, H.D. and Gupta, M., *Problem 343*, Jber. Dt. Math. Verein, **57**(1954), 8-9.

[2] Deaconescu, M. and Sándor, J., *Variations on a theme by Hurwitz*, Gazeta Mat. A, **8**(1987), no. 4, 186-191.

[3] Dickson, L.E., *History of the theory of numbers*, vol. I, Chelsea Publ. Comp., 1919.

[4] Minculete, N. and Dicu, P., *Concerning the Euler totient*, General Mathematics, **16**(2008), no. 1, 93-99.

[5] Sándor, J., *Câteva ecuaţii diofantice pentru funcţii aritmetice particulare*, Semin. t. structurilor, no. 53, pp. 1-10, Univ. Timişoara, 1989.

[6] Sándor, J., *Geometric theorems, diophantine equations and arithmetic functions*, New Mexico, 2002.

[7] Sándor, J. and Crstici, B., *Handbook of number theory II*, Springer Verlag, 2005.

[8] Sándor, J., Mitronović, D.S., and Crstici, B., *Handbook of number theory I*, Springer Verlag, 2006 (first ed., 1996 by Kluwer Acad. Publ.).

[9] Sándor, J., *A stronger lower bound for the σ-function*, to appear in Octogon Math. Mag.

[10] Sándor, J., *Selected chapters of geometry, analysis and number theory: Classical topics in new perspectives*, Lambert Acad. Publ., 2009.

József Sándor
Babeş-Bolyai University
Department of Mathematics
Str. Kogălniceanu Nr. 1
400084 Cluj-Napoca, Romania
E-mails: jsandor@math.ubbcluj.ro, jsandor@hotmail.com