ANOTHER NEW PROOF METHOD OF ANALYTIC INEQUALITY

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Abstract. This paper gives another new proof method of analytic inequality involving n variables. As its applications, we give proof of some known-well inequalities and prove five new analytic inequalities.

1. MONOTONICITY ON SPECIAL VARIABLES

Throughout the paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ denotes the set of strictly positive real numbers. Let $n \geq 2$, $n \in \mathbb{N}$. The arithmetic mean $A(\boldsymbol{x})$ and the power mean $M_r(\boldsymbol{x})$ of order r with respect to the positive real numbers $x_1, x_2, \cdots, x_n$ are defined respectively as $A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$, $M_r(\boldsymbol{x}) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{1/r}$ for $r \neq 0$, and $M_0(\boldsymbol{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$.

In paper [4], the author puts up a new proof method of analytic inequality. In this section, we shall provide another new proof method of analytic inequality involving n variables.

Lemma 1.1. *Let interval $I = [m, M] \subset \mathbb{R}$, $D \overset{def.}{=} \{(x_1, x_2)\,|m \leq x_2 \leq x_1 \leq M\} \subset I^2$ and function $f : I^2 \to \mathbb{R}$ have continuous partial derivative. Then $\partial f/\partial x_1 \geq (\leq)\, \partial f/\partial x_2$ hold in D, if and only if $f(a, b) \geq (\leq)\, f(a - l, b + l)$ hold for all $a, b \in I$ and l with $b < b + l \leq a - l < a$.*

Proof. Without the losing of generality, we only prove the case $\partial f/\partial x_1 \geq \partial f/\partial x_2$.

For all $x_1, x_2 \in D$ and $l \in \mathbb{R}_+$ with $m \leq x_2 < x_2+l \leq x_1-l < x_1 \leq M$, we have $f(x_1 - l, x_2 + l) - f(x_1, x_2) \leq 0$. Then it exists $\xi_l \in (0, l)$ such that

$$l\left(-\frac{\partial f(x_1 - \xi_l, x_2 + \xi_l)}{\partial x_1} + \frac{\partial f(x_1 - \xi_l, x_2 + \xi_l)}{\partial x_2}\right) \leq 0,$$

$$-\frac{\partial f(x_1 - \xi_l, x_2 + \xi_l)}{\partial x_1} + \frac{\partial f(x_1 - \xi_l, x_2 + \xi_l)}{\partial x_2} \leq 0.$$

Let $l \to 0+$, we get

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \geq \frac{\partial f(x_1, x_2)}{\partial x_2}.$$

According to continuity of partial derivative, we know

$$\frac{\partial f(x_1, x_1)}{\partial x_1} \geq \frac{\partial f(x_1, x_1)}{\partial x_2}.$$

hold also.

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On the other hand, assumes $\partial f/\partial x_1 \geq \partial f/\partial x_2$ hold in D. For all $a, b \in I$ and l with $b < b+l \leq a-l < a$, it exists $\xi_l \in (0,l)$ such that

$$\begin{aligned} f(a,b) - f(a-l,b+l) &= -\left(f(a-l,b+l) - f(a,b)\right) \\ &= -l\left(-\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_1} + \frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_2}\right) \\ &= l\left(\frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_1} - \frac{\partial f(a-\xi_l,b+\xi_l)}{\partial x_2}\right) \\ &\geq 0. \end{aligned}$$

we complete the proof of lemma. □

Theorem 1.1. ***(Compressed independent variables theorem)*** *Suppose that $D \subset \mathbb{R}^n$ be a symmetric with respect to permutations and convex set, and it have a nonempty interior set D^0, function $f : D \to \mathbb{R}$ and its partial derivatives be continue, and*

$$\breve{D}_i = \left\{\boldsymbol{x} \in D | x_i = \max_{1\leq j\leq n}\{x_j\}\right\} - \{\boldsymbol{x} \in D | x_1 = x_2 = \cdots = x_n\}, \tag{1.1}$$

$$\widehat{D}_i = \left\{x \in D | x_i = \min_{1\leq j\leq n}\{x_j\}\right\} - \{x \in D | x_1 = x_2 = \cdots = x_n\}, \tag{1.2}$$

$i = 1, 2, \cdots, n$. For all $i, j = 1, 2, \cdots, n, i \neq j$,

$$\partial f/\partial x_i > (<)\, \partial f/\partial x_j$$

in $\breve{D}_i \cap \widehat{D}_j$. Then

$$f(a_1, a_2, \cdots, a_n) \geq (\leq)\, f(A(\boldsymbol{a}), A(\boldsymbol{a}), \cdots, A(\boldsymbol{a})) \tag{1.3}$$

for all $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in D$, equality hold if only if $a_1 = a_2 = \cdots = a_n$.

Proof. If $n = 2$, Let $l \to |a_1 - a_2|/2$ in Lemma 1.1, we complete the proof of theorem. We suppose $n \geq 3$. Without the losing of generality, we only prove the case $\partial f/\partial x_i > \partial f/\partial x_j$ with $i \neq j$.

If $a_1 = a_2 = \cdots = a_n$, the inequality (1.3) hold obviously. If $\max\limits_{1\leq j\leq n}\{a_j\} \neq \min\limits_{1\leq j\leq n}\{a_j\}$, we let $a_1 = \max\limits_{1\leq j\leq n}\{a_j\}$ and $a_n = \min\limits_{1\leq j\leq n}\{a_j\}$.

(1) If $a_1 > \max\limits_{2\leq j\leq n}\{a_j\}$, $a_n < \min\limits_{1\leq j\leq n-1}\{a_j\}$, because $\partial f/\partial x_i > \partial f/\partial x_j$ hold in $\breve{D}_1 \cap \widehat{D}_n$, According to Lemma 1.1, exist $a_1^{(1)}$, $a_n^{(1)}$ such that $l = a_1 - a_1^{(1)} = a_n^{(1)} - a_n > 0$ and $a_1^{(1)} = a_{i_0} = \max\limits_{2\leq j\leq n-1}\{a_j\}$ (let $a_1^{(1)} = a_2$ briefly), or $a_n^{(1)} = a_{j_0} = \min\limits_{2\leq j\leq n-1}\{a_j\}$ (let $a_n^{(1)} = a_{n-1}$ briefly), we have

$$f(a_1, a_2, a_3, \cdots, a_n) \geq f\left(a_1^{(1)}, a_2, a_3, \cdots, a_n^{(1)}\right).$$

Simply, we denote $a_i^{(1)} = a_i, 2 \leq i \leq n-1$. Consequently,

$$f(a_1, a_2, a_3, \cdots, a_n) \geq f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \cdots, a_n^{(1)}\right).$$

If $a_1^{(1)} = a_2^{(1)} = \cdots = a_n^{(1)}$, implies Theorem 1.1 hold. Otherwise, for $a_1^{(1)} = a_2^{(1)} > a_n^{(1)}$, owing to

$$\left.\frac{\partial f(\boldsymbol{x})}{\partial x_1}\right|_{\boldsymbol{x}=\left(a_1^{(1)},a_2^{(1)},a_3^{(1)},\cdots,a_n^{(1)}\right)} > \left.\frac{\partial f(\boldsymbol{x})}{\partial x_n}\right|_{\boldsymbol{x}=\left(a_1^{(1)},a_2^{(1)},a_3^{(1)},\cdots,a_n^{(1)}\right)},$$

and the continuity of partial derivatives, it exists $\varepsilon > 0$ such that

$$\left.\frac{\partial f(\boldsymbol{x})}{\partial x_1}\right|_{\boldsymbol{x}=\left(s,a_2^{(1)},a_3^{(1)},\cdots,t\right)} > \left.\frac{\partial f(\boldsymbol{x})}{\partial x_n}\right|_{\boldsymbol{x}=\left(s,a_2^{(1)},a_3^{(1)},\cdots,t\right)},$$

where $s \in \left[a_1^{(1)} - \varepsilon, a_1^{(1)}\right]$ $t \in \left[a_n^{(1)}, a_n^{(1)} + \varepsilon\right]$. Denote $a_1^{(2)} = a_1^{(1)} - \varepsilon$, $a_n^{(2)} = a_n^{(1)} + \varepsilon$, $a_i^{(2)} = a_i^{(1)} (2 \le i \le n-1)$. By Lemma 1.1, we get

$$f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \cdots, a_n^{(1)}\right) \ge f\left(a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, \cdots, a_n^{(2)}\right), \tag{1.4}$$

and $a_2^{(2)} = \max\limits_{1\le i\le n}\left\{a_i^{(2)}\right\}$. For $a_1^{(1)} > a_{n-1}^{(1)} = a_n^{(1)}$, after a similar argument, we get inequality (1.4) and $a_{n-1}^{(2)} = \min\limits_{1\le i\le n}\left\{a_i^{(2)}\right\}$.

Repeated the above steps, we get $\left\{a_1^{(i)}, a_2^{(i)}, \cdots, a_n^{(i)}\right\}$ $(i = 1, 2, \cdots)$, $\sum_{j=1}^n a_j^{(i)}$ are constant, $\left\{a_j^{(i)}\right\}$ $(i = 1, 2, \cdots)$ are monotone increasing (decreasing) sequences if $a_j \ge (\le) A(\boldsymbol{a}), j = 1, 2, 3, \cdots, n$, and

$$f\left(a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \cdots, a_n^{(1)}\right) \ge f\left(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, \cdots, a_n^{(i)}\right).$$

If exists $i \in \mathbb{N}$, $a_1^{(i)} = a_2^{(i)} = \cdots = a_n^{(i)}$, we complete the proof of theorem. Otherwise, let $\alpha = \inf\limits_{i\in\mathbb{N}}\left\{\max\left\{a_1^{(i)}, a_2^{(i)}, \cdots, a_n^{(i)}\right\}\right\}$. Without the losing of generality, we suppose

$$\max\left\{a_1^{(i_j)}, a_2^{(i_j)}, \cdots, a_n^{(i_j)}\right\} = a_1^{(i_j)} \to \alpha$$

and

$$\lim_{j\to+\infty}\left(a_1^{(i_j)}, a_2^{(i_j)}, \cdots, a_n^{(i_j)}\right) = (\alpha, b_2, b_3, \cdots, b_n),$$

where $\{i_j\}_{j=1}^{+\infty}$ is a subsequence of $\mathbb{N}$. Because of the continuity of function f, it have

$$f(a_1, a_2, a_3, \cdots, a_n) \ge f(\alpha, b_2, b_3, \cdots, b_n).$$

If $\alpha \ne \min\{b_2, b_3, \cdots, b_n\}$, we can repeat the above arguments, this contradicts with the definition of α. Then $\alpha = b_2 = b_3 = \cdots = b_n$.Owing to $\alpha + \sum_{i=2}^n b_i = \sum_{i=1}^n a_i$, we have $\alpha = b_2 = b_3 = \cdots = b_n = A(a)$, the proof of Theorem 1.1 is completed.

(2) For the case $a_1 = \max\limits_{2\le j\le n}\{a_j\}$, or $a_n = \min\limits_{1\le j\le n-1}\{a_j\}$, it have been proved in (1). □

In particular, according to Theorem 1.1 the following corollary hold.

Corollary 1.1. *Suppose that $D \subset \mathbb{R}^n$ is a symmetric with respect to permutations and convex set, and it has a nonempty interior set D^0, function $f : D \to \mathbb{R}$ is symmetric, f and its partial derivatives is continue. Let*

$$\breve{D}_1 = \left\{\boldsymbol{x} \in D | x_1 = \max_{1\le j\le n}\{x_j\}\right\} - \{\boldsymbol{x} \in D | x_1 = x_2 = \cdots = x_n\},$$

$$\widehat{D}_2 = \left\{\boldsymbol{x} \in D | x_2 = \min_{1\le j\le n}\{x_j\}\right\} - \{\boldsymbol{x} \in D | x_1 = x_2 = \cdots = x_n\},$$

$$D^* = \breve{D}_1 \cap \widehat{D}_2. \tag{1.5}$$

If $\partial f/\partial x_1 > (<) \, \partial f/\partial x_2$ hold in D^*, then

$$f(a_1, a_2, \cdots, a_n) \ge (\le) f(A(\boldsymbol{a}), A(\boldsymbol{a}), \cdots, A(\boldsymbol{xa}))$$

for all $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in D$, equality hold if only if $a_1 = a_2 = \cdots = a_n$.

2. Unifying Proof of Some Well-known Inequality

Take advantage of Theorem 1.1 and Corollary 1.1, we can prove some well-known inequality, for example, Power Mean Inequality, Holder-Inequality, Minkowski-Inequality. In this section, we only prove an example.

Proposition 2.1. *(Holder-Inequality) Let* $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$, $p, q > 1$, *and* $1/p + 1/q = 1$. *Then*

$$\left(\sum_{k=1}^n x_k^p\right)^{1/p} \left(\sum_{k=1}^n y_k^q\right)^{1/q} \geq \sum_{k=1}^n x_k y_k.$$

Proof. Let $(a_1, a_2, \cdots, a_n) \in \mathbb{R}_+^n$ and function

$$f : \boldsymbol{b} \to \left(\sum_{k=1}^n a_k\right)^{1/p} \left(\sum_{k=1}^n a_k b_k\right)^{1/q} - \sum_{k=1}^n a_k b_k^{1/q}, \quad b \in \mathbb{R}_+^n.$$

We have

$$\frac{\partial f}{\partial b_i} = 1/q \cdot \left(\sum_{k=1}^n a_k\right)^{1/p} \left(\sum_{k=1}^n a_k b_k\right)^{1/q-1} a_i - 1/q \cdot a_i b_i^{1/q-1},$$

$$\frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} = \frac{1}{q}\left(\frac{\sum_{k=1}^n a_k}{\sum_{k=1}^n b_k a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right).$$

Let $\boldsymbol{b} \in \breve{D}_i \cap \widehat{D}_j$ (see (1.1) and (1.2)). (1)If $a_i \geq a_j$,

$$\begin{aligned}\frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} &\geq \frac{1}{q}\left(\frac{\sum_{k=1}^n a_k}{b_i \sum_{k=1}^n a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right) \\ &= \frac{1}{q} a_j \left(b_j^{-1/p} - b_i^{-1/p}\right) > 0.\end{aligned}$$

(2)If $a_i \leq a_j$,

$$\begin{aligned}\frac{\partial f}{\partial b_i} - \frac{\partial f}{\partial b_j} &\geq \frac{1}{q}\left(\frac{\sum_{k=1}^n a_k}{b_j \sum_{k=1}^n a_k}\right)^{1/p} (a_i - a_j) - \frac{1}{q}\left(a_i b_i^{-1/p} - a_j b_j^{-1/p}\right) \\ &= \frac{1}{q} a_i \left(b_j^{-1/p} - b_i^{-1/p}\right) > 0.\end{aligned}$$

According to 1.1, we get

$$f(\boldsymbol{b}) \geq f\left(A(\boldsymbol{b}), A(\boldsymbol{b}), \cdots, A(\boldsymbol{b})\right).$$

that is

$$\left(\sum_{k=1}^n a_k\right)^{1/p} \left(\sum_{k=1}^n a_k b_k\right)^{1/q} - \sum_{k=1}^n a_k b_k^{1/q} \geq 0. \tag{2.1}$$

Let $a_k = x_k^p, , b_k = y_k^q / x_k^p$ in (2.1), we know Holder-Inequality hold. □

3. Five new inequalities

Let $n \geq 3$, $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}_+^n$, $\prod_n^k(\boldsymbol{a}) = \left(\prod_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k} \sum_{j=1}^k a_{i_j}\right)^{1 \Big/ \binom{n}{k}}$ is the third symmetric mean of $\boldsymbol{a}$ (see [2]).

Theorem 3.1. *Let* $2 \leq k \leq n-1$, $p = (k-1)/(n-1)$. *Then*

$$\prod_n^k(\boldsymbol{a}) \geq [A(\boldsymbol{a})]^p [M_0(\boldsymbol{a})]^{1-p}, \tag{3.1}$$

$p = (k-1)/(n-1)$ *is the best constant.*

Proof. Suppose

$$f(\boldsymbol{a}) = \left[\prod_{i=1}^{n} a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}} \cdot \prod_{1\le i_1<\cdots<i_k\le n} \frac{1}{k}\sum_{j=1}^{k} a_{i_j}, \quad a=(a_1,a_2,\cdots,a_n)\in\mathbb{R}_+^n.$$

Then

$$\frac{\partial f}{\partial a_1} = -\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1}\left[\prod_{i=1}^{n} a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}} \cdot \prod_{1\le i_1<\cdots<i_k\le n} \frac{1}{k}\sum_{j=1}^{k} a_{i_j},$$

(3.2)

$$\begin{aligned}\frac{\partial f}{\partial a_1}-\frac{\partial f}{\partial a_2} &= -\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}\left[\prod_{i=1}^{n} a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}} \cdot \prod_{1\le i_1<\cdots<i_k\le n} \frac{1}{k}\sum_{j=1}^{k} a_{i_j}\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\\ &= \frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2}\left[\prod_{i=1}^{n} a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}} \cdot \prod_{1\le i_1<\cdots<i_k\le n} \frac{1}{k}\sum_{j=1}^{k} a_{i_j}(a_1-a_2)\\ &\quad -\left[\prod_{i=1}^{n} a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}} \cdot \prod_{1\le i_1<\cdots<i_k\le n} \frac{1}{k}\sum_{j=1}^{k} a_{i_j}\\ &\quad \cdot \sum_{3\le i_1<\cdots i_{k-1}\le n} \frac{a_1-a_2}{\left(a_1+\sum_{j=1}^{k-1} a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1} a_{i_j}\right)}.\end{aligned}$$

If $\boldsymbol{a}\in D^*$ (see (1.5)),

$$a_1+(k-1)a_2>a_1,$$

$$\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2} > \frac{\binom{n-2}{k-1}}{ka_2(a_1+(k-1)a_2)},$$

$$\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2} > \sum_{3\le i_1<\cdots i_{k-1}\le n} \frac{1}{(a_1+(k-1)a_2)ka_2},$$

$$\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)a_1a_2} > \sum_{3\le i_1<\cdots i_{k-1}\le n} \frac{1}{\left(a_1+\sum_{j=1}^{k-1} a_{i_j}\right)\left(a_2+\sum_{j=1}^{k-1} a_{i_j}\right)}. \tag{3.3}$$

Combining inequality (3.2) to inequality (3.3), we have $\partial f/\partial a_1-\partial f/\partial a_2>0$. Owing to Corollary 1.1, we get

$$f(a_1,a_2,\cdots,a_n)\ge(\le)f(A(\boldsymbol{a}),A(\boldsymbol{a}),\cdots,A(\boldsymbol{a}))$$

for all $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}_+^n$. It implies

$$\left[\prod_{i=1}^n a_i\right]^{-\frac{(n-k)\cdot\binom{n}{k}}{n(n-1)}}\cdot\prod_{1\le i_1<\cdots<i_k\le n}k^{-1}\sum_{j=1}^k a_{i_j}\ge[A(\boldsymbol{a})]^{\frac{(k-1)\cdot\binom{n}{k}}{n-1}}.$$

Inequality (3.1) is proved.

Let $a_1=a_2=\cdots=a_{n-1}=1, a_n=x$ in inequality (3.1), it lead to

$$\left(\frac{x+k-1}{k}\right)^{\binom{n-1}{k-1}\Big/\binom{n}{k}}\ge\left(\frac{x+n-1}{n}\right)^p\left(\sqrt[n]{x}\right)^{1-p}.$$

$$p\le\frac{\frac{k}{n}\ln\frac{x+k-1}{k}-\frac{1}{n}\ln x}{\ln(x+n-1)-\ln n\sqrt[n]{x}}.$$

Let $x\to+\infty$ in above inequality,

$$p\le\lim_{x\to+\infty}\frac{\frac{k}{n}\cdot\frac{1}{x+k-1}-\frac{1}{nx}}{\frac{1}{x+n-1}-\frac{1}{nx}}=\lim_{x\to+\infty}\frac{\frac{kx}{x+k-1}-1}{\frac{nx}{x+n-1}-1}=\frac{k-1}{n-1}.$$

So $p=(k-1)/(n-1)$ is the best constant. □

Theorem 3.2. *Suppose $n\ge3$, $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}_+^n$, $\beta>0>\alpha$. If $\beta+\alpha>0$, let $\lambda=\frac{-2\alpha}{n(\beta-\alpha)}$. If $\beta+\alpha\le0$, let $\lambda=\frac{1}{n}$. Then*

$$[M_\alpha(\boldsymbol{a})]^{1-\lambda}\cdot[M_\beta(\boldsymbol{a})]^\lambda\le M_0(\boldsymbol{a}). \tag{3.4}$$

Proof. Let $f(\boldsymbol{x})=\frac{1}{n\beta}\ln\left(\prod_{i=1}^n x_i\right)-\frac{1-\lambda}{\alpha}\ln\left(\frac{1}{n}\sum_{i=1}^n x_i^{\alpha/\beta}\right)$, $\boldsymbol{x}\in\mathbb{R}_+^n$. Then

$$\frac{\partial f(\boldsymbol{x})}{\partial x_j}=\frac{1}{n\beta x_j}-\frac{1-\lambda}{\beta}\frac{x_j^{\alpha/\beta-1}}{\sum_{i=1}^n x_i^{\alpha/\beta}},\ j=1,2,$$

$$\frac{\partial f(\boldsymbol{x})}{\partial x_1}-\frac{\partial f(\boldsymbol{x})}{\partial x_2}=\frac{x_2-x_1}{n\beta x_1x_2}-\frac{1-\lambda}{\beta}\frac{x_1^{\alpha/\beta-1}-x_2^{\alpha/\beta-1}}{\sum_{i=1}^n x_i^{\alpha/\beta}}. \tag{3.5}$$

Case 1 :$\alpha+\beta>0$. Let

$$g(t)=\frac{\beta+\alpha}{\beta-\alpha}t^{\beta-\alpha}-t^\beta+t^{-\alpha}-\frac{\beta+\alpha}{\beta-\alpha},\quad t\in(1,+\infty).$$

Then

$$t^{\alpha+1}g'(t)=(\beta+\alpha)t^\beta-\beta t^{\beta+\alpha}-\alpha,$$
$$\left[t^{\alpha+1}g'(t)\right]'=(\beta+\alpha)\beta t^{\beta+\alpha-1}\left(t^{-\alpha}-1\right)>0.$$

Therefore $t^{\alpha+1}g'(t)$ is monotone increasing function in $(1,+\infty)$. Meanwhile

$$\lim_{t\to1+}t^{\alpha+1}g'(t)=\lim_{t\to1+}\left[(\beta+\alpha)t^\beta-\beta t^{\beta+\alpha}-\alpha\right]=0.$$

Thence, $t^{\alpha+1}g'(t)>0$, $g'(t)>0$. In addition to $\lim_{t\to1+}g(t)=0$, we have $g(t)>0$ and

$$\frac{\beta+\alpha}{\beta-\alpha}t^{\beta-\alpha}-t^\beta+t^{-\alpha}-\frac{\beta+\alpha}{\beta-\alpha}>0$$

$$\frac{\beta+\alpha}{\beta-\alpha}t^\beta-\left(1+\frac{2\alpha}{\beta-\alpha}\right)t^\alpha-t^{\alpha+\beta}+1>0,$$

$$\frac{\beta+\alpha}{\beta-\alpha}t^{\beta}-\left(n-1+\frac{2\alpha}{\beta-\alpha}\right)t^{\alpha}-t^{\alpha+\beta}+(n-1)>0,$$

$$(1-n\lambda)\,t^{\beta}-(n-1-n\lambda)\,t^{\alpha}-t^{\alpha+\beta}+(n-1)>0, \tag{3.6}$$

$$(1-\lambda)\,\frac{1-t^{\alpha-\beta}}{t^{\alpha}+(n-1)}>\frac{t^{\beta}-1}{nt^{\beta}}. \tag{3.7}$$

We assume that $x\in D^*$ (see (1.5)), let $t=(x_1/x_2)^{1/\beta}$ in above inequality. It has

$$(1-\lambda)\,\frac{x_2^{\alpha/\beta-1}-x_1^{\alpha/\beta-1}}{x_1^{\alpha/\beta}+(n-1)\,x_2^{\alpha/\beta}}>\frac{x_1-x_2}{nx_1x_2},$$

$$\frac{1-\lambda}{\beta}\frac{x_2^{\alpha/\beta-1}-x_1^{\alpha/\beta-1}}{\sum_{i=1}^{n}x_i^{\alpha/\beta}}>\frac{x_1-x_2}{n\beta x_1x_2}. \tag{3.8}$$

Combining inequality (3.5) to inequality (3.8), we have $\partial f\left(\boldsymbol{x}\right)/\partial x_1-\partial f\left(v\right)/\partial x_2>0$. According to Corollary 1.1, we get

$$f\left(x_1,x_2,\cdots,x_n\right)\geq f\left(A\left(\boldsymbol{x}\right),A\left(\boldsymbol{x}\right),\cdots,A\left(\boldsymbol{x}\right)\right),$$

$$\frac{1}{n\beta}\ln\left(\prod\nolimits_{i=1}^{n}x_i\right)-\frac{1-\lambda}{\alpha}\ln\left(\frac{1}{n}\sum\nolimits_{i=1}^{n}x_i^{\alpha/\beta}\right)\geq\frac{\lambda}{\beta}\ln\left(\frac{1}{n}\sum\nolimits_{i=1}^{n}x_i\right).$$

Let $a_i=x_i^{1/\beta},i=1,2,\cdots,n$, it get

$$\left[M_{\alpha}\left(\boldsymbol{a}\right)\right]^{1-\lambda}\cdot\left[M_{\beta}\left(\boldsymbol{a}\right)\right]^{\lambda}\leq M_0\left(\boldsymbol{a}\right).$$

Case 2: $\alpha+\beta<0$. Let $t>1$, because $\alpha<0,\alpha+\beta<0$, it has

$$(n-1)>(n-2)\,t^{\alpha}+t^{\alpha+\beta}.$$

We find that inequality (3.7) hold.The rest of the proofs are similar, so we shall omit them. □

A similar argument lead to Theorem 3.3

Theorem 3.3. *Suppose $n\geq 3$, $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}_+^n$, $\beta>0>\alpha$. If $\beta+\alpha>0$, let $\theta=\frac{n-1}{n}$. If $\beta+\alpha\leq 0$, let $\theta=1-\frac{2\beta}{n(\beta-\alpha)}$. Then*

$$M_0\left(\boldsymbol{a}\right)\leq\left[M_{\alpha}\left(\boldsymbol{a}\right)\right]^{1-\theta}\cdot\left[M_{\beta}\left(\boldsymbol{a}\right)\right]^{\theta}. \tag{3.9}$$

Remark 3.1. *Inequality 3.4 3.9 improve the well-known Sierpinski-Inequality*

$$\left[M_{-1}\left(\boldsymbol{a}\right)\right]^{\frac{n-1}{n}}\left[A\left(\boldsymbol{a}\right)\right]^{\frac{1}{n}}\leq M_0\left(\boldsymbol{a}\right)\leq\left[M_{-1}\left(\boldsymbol{a}\right)\right]^{\frac{1}{n}}\left[A\left(\boldsymbol{a}\right)\right]^{\frac{n-1}{n}}.$$

For $n\geq 2$, $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}_+^n$, paper [1] introduces an inequality

$$\frac{n-1}{n}A\left(\boldsymbol{a}\right)+\frac{1}{n}M_{-1}\left(\boldsymbol{a}\right)\geq M_0\left(\boldsymbol{a}\right)$$

We can improve the inequality by Theorem 3.4 and Theorem 3.5.

Theorem 3.4. *Suppose $p=n^2/\left(n^2+4n-4\right)$, then*

$$pA\left(\boldsymbol{a}\right)+(1-p)\,M_{-1}\left(\boldsymbol{a}\right)\geq M_0\left(\boldsymbol{a}\right). \tag{3.10}$$

Proof. Firstly, Let $p > n^2 \big/ \left(n^2+4n-4\right)$,

$$f\left(\boldsymbol{b}\right) = p/n \cdot \sum\nolimits_{i=1}^{n} e^{b_i} + \left(1-p\right) n \cdot \left(\sum\nolimits_{i=1}^{n} e^{-b_i}\right)^{-1}, \quad b = \left(b_1, b_2, \cdots, b_n\right) \in \mathbb{R}^n.$$

Then

$$\frac{\partial f}{\partial b_1} = \frac{p}{n} e^{b_1} + \left(1-p\right) \frac{n}{\left(\sum_{i=1}^{n} e^{-b_i}\right)^2} e^{-b_1},$$

$$\frac{\partial f}{\partial b_1} - \frac{\partial f}{\partial b_2} = \frac{p}{n}\left(e^{b_1} - e^{b_2}\right) - \left(1-p\right) \frac{n}{\left(\sum_{i=1}^{n} e^{-b_i}\right)^2} \left(e^{-b_2} - e^{-b_1}\right).$$

If $b_1 = \max\limits_{1\le i\le n}\{b_i\} > b_2 = \min\limits_{1\le i\le n}\{b_i\}$, $t = e^{b_1-b_2} > 1$. We have

$$\begin{aligned}
\frac{\partial f}{\partial b_1} - \frac{\partial f}{\partial b_2} &\ge \frac{p}{n}\left(e^{b_1} - e^{b_2}\right) - \left(1-p\right) \frac{n}{\left(\left(n-1\right)e^{-b_1} + e^{-b_2}\right)^2}\left(e^{-b_2} - e^{-b_1}\right) \\
&= \frac{\left(e^{b_1} - e^{b_2}\right)}{n\left(\left(n-1\right)e^{b_2} + e^{b_1}\right)^2}\left[p\left(\left(n-1\right)e^{b_2} + e^{b_1}\right)^2 - \left(1-p\right) n^2 e^{b_1} e^{b_2}\right] \\
&= \frac{e^{3b_2}\left(t-1\right)}{n\left(\left(n-1\right)e^{b_2} + e^{b_1}\right)^2}\left[p\left(n-1+t\right)^2 - n^2 t + p n^2 t\right]. \\
&> \frac{e^{3b_2}\left(t-1\right)}{n\left(\left(n-1\right)e^{b_2} + e^{b_1}\right)^2}\left[\frac{n^2}{n^2+4n-4}\left(n-1+t\right)^2 - n^2 t + \frac{n^2}{n^2+4n-4} n^2 t\right] \\
&= \frac{n e^{3b_2}\left(t-1\right)\left(t-n+1\right)^2}{\left(n^2+4n-4\right)\left(\left(n-1\right)e^{b_2} + e^{b_1}\right)^2} \ge 0.
\end{aligned}$$

According to Corollary 1.1, we get

$$f\left(\boldsymbol{b}\right) \ge f\left(A\left(\boldsymbol{b}\right), A\left(b\right), \cdots, A\left(\boldsymbol{b}\right)\right),$$

$$\frac{p}{n}\sum\nolimits_{i=1}^{n} e^{b_i} + \left(1-p\right)\frac{n}{\sum_{i=1}^{n} e^{-b_i}} \ge e^{A(b)} = \sqrt[n]{\prod\nolimits_{i=1}^{n} e^{b_i}}.$$

Let $e^{b_i} = a_i$ in above inequality, we know inequality (3.10) hold. Because of continuity, if $p = n^2 \big/ \left(n^2+4n-4\right)$, inequality (3.10) hold also . □

Theorem 3.5. *Suppose* $p = \left(1 - n - \sqrt{5n^2-6n+1}\right) \big/ 2n$*, then*

$$\frac{n-1}{n} A\left(\boldsymbol{a}\right) + \frac{1}{n} M_p\left(\boldsymbol{a}\right) \ge M_0\left(\boldsymbol{a}\right).$$

Proof. Let

$$f\left(\boldsymbol{a}\right) = \sqrt[n]{\prod\nolimits_{i=1}^{n} a_i^{1/p}} - \left(n-1\right)/n^2 \cdot \sum\nolimits_{i=1}^{n} a_i^{1/p}, \quad \boldsymbol{a} \in \mathbb{R}_+^n.$$

Then

$$\frac{\partial f}{\partial a_1} = \frac{1}{npa_1}\sqrt[n]{\prod\nolimits_{i=1}^{n} a_i^{1/p}} - \frac{n-1}{n^2 p} a_1^{1/p-1},$$

$$\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} = -\frac{a_1 - a_2}{npa_1a_2}\prod\nolimits_{i=1}^{n} a_i^{1/np} - \frac{n-1}{n^2 p}\left(a_1^{1/p-1} - a_2^{1/p-1}\right).$$

If $a_1 = \max\limits_{1\le i\le n}\{a_i\} > a_2 = \min\limits_{1\le i\le n}\{a_i\} > 0$, $a_1/a_2 = t > 1$. Owing to $p < 0, -\frac{a_1-a_2}{npa_1a_2} > 0$, we have

$$\begin{aligned}
\frac{\partial f}{\partial a_1} - \frac{\partial f}{\partial a_2} &\le -\frac{a_1 - a_2}{npa_1a_2} a_1^{1/(np)} a_2^{(n-1)/(np)} - \frac{n-1}{n^2 p}\left(a_1^{1/p-1} - a_2^{1/p-1}\right) \\
&= \frac{a_1^{1/p-1}}{n^2 p}\left[-n\frac{t-1}{t} t^{1-(n-1)/(np)} - \left(n-1\right)\left(1 - t^{1-1/p}\right)\right].
\end{aligned} \tag{3.11}$$

Let $g(t) = -nt^{1-(n-1)/(np)} + nt^{-(n-1)/(np)} + (n-1)\,t^{1-1/p} - (n-1)\,, \quad t > 1.$

$$g'(t) = \left(-n + \frac{n-1}{p}\right) t^{-(n-1)/(np)} - \frac{n-1}{p} t^{-1-(n-1)/(np)} + (n-1)\left(1 - \frac{1}{p}\right) t^{-1/p},$$

$$t^{1+(n-1)/(np)} g'(t) = \left(-n + \frac{n-1}{p}\right) t - \frac{n-1}{p} + (n-1)\left(1 - \frac{1}{p}\right) t^{1-1/(np)}.$$

$$\begin{aligned}\left(t^{1+(n-1)/(np)} g'(t)\right)' &= \left(-n + \frac{n-1}{p}\right) + (n-1)\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{np}\right) t^{-1/(np)} \\ &> \left(-n + \frac{n-1}{p}\right) + (n-1)\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{np}\right) = 0.\end{aligned}$$

Thus $t^{1+(n-1)/(np)} g'(t)$ is a monotone increasing function. Meanwhile,

$$\begin{aligned}\lim_{t\to 1+} t^{1+(n-1)/(np)} g'(t) &= \lim_{t\to 1+} \left[\left(-n + \frac{n-1}{p}\right) t - \frac{n-1}{p} + (n-1)\left(1 - \frac{1}{p}\right) t^{1-\frac{1}{np}}\right] \\ &= -1 - \frac{n-1}{p} \geq 0.\end{aligned}$$

Therefore $t^{1+(n-1)/(np)} g'(t) > 0$, $g'(t) > 0$, $g(t)$ is monotone increasing function. Meanwhile, $\lim\limits_{t\to 1+} g(t) = 0$, then $g(t) > 0$. By (3.11), we know $\partial f/\partial a_1 - \partial f/\partial a_2 < 0$. According to Corollary 1.1,

$$f(\boldsymbol{a}) \leq f(A(\boldsymbol{a}), A(\boldsymbol{a}), \cdots, A(\boldsymbol{a}))\,,$$

$$\sqrt[n]{\prod\nolimits_{i=1}^{n} a_i^{1/p}} - \frac{n-1}{n^2} \cdot \sum\nolimits_{i=1}^{n} a_i^{1/p} \leq \frac{1}{n} \cdot A^{1/p}(\boldsymbol{a})$$

Finally, let $a_i \to a_i^p\,(i = 1, 2, \cdots, n)$ in the above inequality, we know Theorem 3.5 hold. □

Remark 3.2. *More applications of Theorem 1.1 and Corollary 1.1 will appear in book* [5] *and http://old.irgoc.org/Article/ShowArticle.asp?ArticleID=391.*

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