# SOME INEQUALITIES FOR GRAMIAN NORMAL OPERATORS AND SEMINORMS IN PSEUDO-HILBERT SPACES

LOREDANA CIURDARIU

Abstract. Two inequalities are presented, at first for the moduli of the gramian normal operators, being a form of the Cauchy-Bunyakovsky-Schwarz inequality obtained in [6], Theorem 1 and the second being a form of Theorem 5, see [5] for a particular seminorm.

## 1. Introduction

Let $Z$ an admissible space in the Loynes sense, see [8] and $\mathcal{H}$ be a Loynes $Z$-space. As in Hilbert spaces, see [4], we can show also that in pseudo-Hilbert spaces for any two operators $T, U \in \mathcal{B}^*(\mathcal{H})$, see [2], we have

$$||TU - UT|| \leq 2\min\{|T||, ||U|||\}\min\{||T-U||, ||T+U||\}.$$

In addition, if $N$ is a gramian normal operator in $\mathcal{B}^*(\mathcal{H})$ then

$$||NN^* - N^*N|| \leq 2||N||\min\{||N-N^*||, ||N+N^*||\}.$$

If we have three operators, $T_1,\ T_2,\ T_3 \in \mathcal{B}^*(\mathcal{H})$ then

$$||T_1T_2T_3 - T_3T_2T_1|| \leq 2(||T_1||\min\{||T_3||, ||T_2||\}||T_2 - T_3|| +$$

$$+||T_2||\min\{||T_1||, ||T_3||\}||T_3 - T_1|| + ||T_3||\min\{||T_2||, ||T_1||\}||T_2 - T_1||).$$

Indeed,

$$||T_1T_2T_3 - T_3T_2T_1|| = ||T_1(T_2T_3 - T_3T_2) + (T_1T_3 - T_3T_1)T_2 + T_3(T_1T_2 - T_2T_1)|| \leq$$

$$\leq ||T_1||||T_2T_3 - T_3T_2|| + ||T_2||||T_1T_3 - T_3T_1|| + ||T_3||||T_1T_2 - T_2T_1||$$

and now we use first inequality.

The following result is a generalization of Theorem 1, see [6] for gramian normal commuting operators.

**Proposition 1.** *Let $\mathcal{H}$ be a Loynes $Z$-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1,...,n\},\ n \geq 2$ be gramian normal operators which commute, i.e. $N_iN_j = N_jN_i,\ M_iM_j = M_jM_i,\ N_iM_j = M_jN_i,\ (\forall)\ i,j \in \{1,...,n\}$ with the property that $\sum_{k=1}^n N_kM_k = 0$.*
*Then*

$$\max_{i\in\{1,\ldots,n\}}\{|N_iM_i|\} \leq \frac{1}{2}(\sum_{k=1}^n |N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^n |M_k|^2)^{\frac{1}{2}}$$

---

*Date*: December 18, 2009.
2000 *Mathematics Subject Classification.* Primary 47A45; Secondary 42B10.
*Key words and phrases.* pseudo-Hilbert spaces (Loynes spaces), seminorm, Cauchy-Bunyakovski-Schvarz inequality .

*Proof.* The proof will be as in Theorem 1. So for any $i \in \{1, .., n\}$,

$$N_i M_i = -\sum_{k=1, k\neq i}^{n} N_k M_k.$$

Then using the Cauchy-Bunyakovski-Schwarz inequality, the modulus will be,

$$|N_i M_i|^2 = |\sum_{k=1,k\neq i}^{n} N_k M_k|^2 \leq (\sum_{k=1,k\neq i}^{n} |N_k|^2)(\sum_{k=1,k\neq i}^{n} |M_k|^2) =$$

$$= (\sum_{k=1}^{n} |N_k|^2 - |N_i|^2)(\sum_{k=1}^{n} |M_k|^2 - |M_i|^2),$$

for all $i \in \{1, ..., n\}$. We mention that in the proof of the Cauchy-Bunyakovski-Schwarz inequality we can use induction and Fuglede′s theorem.

Also by calculi, the following inequality is true:

$$(A^2 - B^2)(C^2 - D^2) \leq (AC - BD)^2,$$

if $A,\ B,\ C,\ D > 0$ and $A,\ B,\ C,\ D$ are commutative as pairs operatori from $\mathcal{B}^*(\mathcal{H})$.

In our case, we used that if $A,\ B \in \mathcal{B}^*(\mathcal{H}),\ A,\ B \geq 0,\ A^2B^2 = B^2A^2$ then $AB = BA$. Therefore,

$$|N_i M_i|^2 \leq (\sum_{k=1}^{n} |N_k|^2 - |N_i|^2)(\sum_{k=1}^{n} |M_k|^2 - |M_i|^2) =$$

$$= \{[(\sum_{k=1}^{n} |N_k|^2)^{\frac{1}{2}}]^2 - |N_i|^2\}\{[(\sum_{k=1}^{n} |M_k|^2)^{\frac{1}{2}}]^2 - |M_i|^2\} \leq$$

$$\leq [(\sum_{k=1}^{n} |N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^{n} |M_k|^2)^{\frac{1}{2}} - |N_i| \cdot |M_i|]^2 =$$

$$= [(\sum_{k=1}^{n} |N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^{n} |M_k|^2)^{\frac{1}{2}} - |N_i M_i|]^2,$$

$(\forall)\ i \in \{1, ..., n\}$.

Taking into account that the function $f(x) = x^{\frac{1}{2}}$ is operator monotone and that member of our inequality are positive operators in $\mathcal{B}^*(\mathcal{H})$, by [9], will have,

$$|N_i M_i| \leq (\sum_{k=1}^{n} |N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^{n} |M_k|^2)^{\frac{1}{2}} - |N_i M_i|,$$

$(\forall)\ i \in \{1, ..., n\}$ i.e. the conclusion. □

As in [6], Corollary 1 we have,

**Remark 1.** *If $N_k,\ k \in \{1, ..., n\}$ are $n$ gramian normal commutative operators in $\mathcal{B}^*(\mathcal{H})$ and $p_k \geq 0,\ k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then we have,*

$$\max_{i\in\{1,...,n\}} \{p_i |N_i - \sum_{j=1}^{n} p_j N_j|\} \leq \frac{1}{2}(\sum_{k=1}^{n} p_k^2 I)^{\frac{1}{2}} \cdot (\sum_{k=1}^{n} |N_k - \sum_{j=1}^{n} p_j N_j|^2)^{\frac{1}{2}} =$$

$$= \frac{1}{2}(\sum_{k=1}^{n} p_k^2 I)^{\frac{1}{2}} \cdot \{\sum_{k=1}^{n} |N_k|^2 + n|\sum_{j=1}^{n} p_j N_j|^2 - [\sum_{k=1}^{n} N_k^* \cdot \sum_{j=1}^{n} p_j N_j + \sum_{k=1}^{n} N_k \cdot \sum_{j=1}^{n} p_j N_j^*]\}.$$

*Proof.* We will replace as in the proof of Corollary 1, see [6], $N_k$ by $p_k I$ and $M_k$ by $N_k - \sum_{j=1}^n p_j N_j$ which are gramian normal commutative operators too. Then we will apply the inequality from Proposition 1. □

The following results will represent a version of the inequalities from [6] for particular operators.

**Remark 2.** *If* $\min_{i\in\{1,...,n\}} p_i = p_m$ *then by applying the above inequality obtain:*

$$\max_{i\in\{1,...,n\}}\{|N_i - \sum_{j=1}^{n} p_j N_j|\} \leq \frac{1}{2p_m}(\sum_{k=1}^{n} p_k^2 I)^{\frac{1}{2}} \cdot (\sum_{k=1}^{n} |N_k - \sum_{j=1}^{n} p_j N_j|^2)^{\frac{1}{2}} =$$

Now putting in Proposition 1, $\sqrt{p_k}N_k$ instead of $N_k$ and $\sqrt{p_k}M_k$ instead of $M_k$ will have

**Theorem 1.** *Let $\mathcal{H}$ be a Loynes Z-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute, and $p_k \geq 0,\ k \in \{1, ..., n\}$, $\sum_{k=1}^n p_k = 1$ with the property that $\sum_{k=1}^n p_k N_k M_k = 0$.*

*Then*

$$\max_{i\in\{1,...,n\}}\{p_i|N_i M_i|\} \leq \frac{1}{2}(\sum_{k=1}^{n} p_k|N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^{n} p_k|M_k|^2)^{\frac{1}{2}}.$$

If we take in Theorem 1, $M_k = I,\ k \in \{1, ..., n\}$ the above inequality becomes:

$$\max_{i\in\{1,...,n\}}\{p_i|N_i|\} \leq \frac{1}{2}(\sum_{k=1}^{n} p_k|N_k|^2)^{\frac{1}{2}},$$

when $\sum_{k=1}^n p_k N_k = 0$. Because $\sum_{k=1}^n p_k(N_k - \sum_{j=1}^n p_j N_j) = 0$ when $\sum_{k=1}^n p_k = 1$ we can replace above $N_i$ with $N_i - \sum_{j=1}^n p_j N_j$ obtaining the below inequality. The equality in the next Corollary will result from,

$$|N_k - \sum_{j=1}^{n} p_j N_j|^2 = (N_k^* - \sum_{j=1}^{n} p_j N_j^*)(N_k - \sum_{j=1}^{n} p_j N_j) = |N_k|^2 + |\sum_{j=1}^{n} p_j N_j|^2 -$$

$-N_k^* \sum_{j=1}^n p_j N_j - \sum_{j=1}^n p_j N_j^* N_k$ by summing after $k$ from 1 to n last equality multiplied by $p_k$.

**Corollary 1.** *With the assumptions in Remark 1, we have that*

$$\max_{i\in\{1,..,n\}}\{p_i|N_i - \sum_{j=1}^{n} p_j N_j|\} \leq \frac{1}{2}(\sum_{k=1}^{n} p_k|N_k - \sum_{j=1}^{n} p_j N_j|^2)^{\frac{1}{2}} =$$

$$= \frac{1}{2}(\sum_{k=1}^{n} p_k|N_k|^2 - |\sum_{j=1}^{n} p_j N_j|^2)^{\frac{1}{2}}.$$

**Proposition 2.** *If we consider $\mathcal{H}$ be a Hilbert space and $N_k, M_k \in \mathcal{B}(\mathcal{H})$ as in Proposition 1, the same inequality will be satisfied.*

The following two results were presented in [3].

**Lemma 1.** *If $p$ is a continuous and monotonous seminorm on $Z$, then $q_p(x) = (p([x,x]))^{1/2}$ is a continuous seminorm on $\mathcal{H}$.*

**Proposition 3.** *If $\mathcal{H}$ is a pre-Loynes $Z$-space and $\mathcal{P}$ is a set of monotonous (increasing) seminorms defining the topology of $Z$, then the topology of $\mathcal{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathcal{P}} = \{q_p \mid p \in \mathcal{P}\}$.*

## 2. The results

The following result was proven for norms on Hilbert spaces in [5] and the proof will follow the same ideas.

**Proposition 4.** *Let $\mathcal{H}$ be a Loynes $Z$ space. If $x_1, ..., x_n \in \mathcal{H}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. If there are constants $r_i > 0$, $i \in \{1, ..., n\}$ such that*

$$q_p(x_i - \sum_{j=1}^n p_j x_j) \leq r_i, \tag{1}$$

*for each $i \in \{1, ..., n\}$, then*

$$p(\sum_{i=1}^n p_i [x_i, x_i] - [\sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j]) \leq \sum_{i=1}^n p_i r_i^2, \tag{2}$$

*or*

$$p(\sum_{i=1}^n p_i [x_i, x_i]) - q_p^2(\sum_{i=1}^n p_i x_i) \leq \sum_{i=1}^n p_i r_i^2. \tag{3}$$

*Proof.* Using the hypothesis (1) we have,

$$q_p^2(x_i - \sum_{j=1}^n p_j x_j) \leq r_i^2$$

for each $i \in \{1, ..., n\}$ and by calculus,

$$p([x_i, x_i] - [x_i, \sum_{j=1}^n p_j x_j] - [\sum_{j=1}^n p_j x_j, x_i] + [\sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j]) \leq r_i^2$$

for each $i \in \{1, ..., n\}$. Multiplying by $p_i \geq 0$ we obtain

$$p(p_i [x_i, x_i] - [p_i x_i, \sum_{j=1}^n p_j x_j] - [\sum_{j=1}^n p_j x_j, p_i x_i] + p_i [\sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j]) \leq p_i r_i^2$$

and summing from $i = 1$ to n we have

$$\sum_{i=1}^n p(p_i [x_i, x_i] - [p_i x_i, \sum_{j=1}^n p_j x_j] - [\sum_{j=1}^n p_j x_j, p_i x_i] + p_i [\sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j]) \leq \sum_{i=1}^n p_i r_i^2.$$

Now by the well-known inequality, $p(y_1 + y_2) \leq p(y_1) + p(y_2)$, $(\forall)\ y_1,\ y_2 \in Z$ it results

$$p(\sum_{i=1}^n (p_i [x_i, x_i] - [p_i x_i, \sum_{j=1}^n p_j x_j] - [\sum_{j=1}^n p_j x_j, p_i x_i] + p_i [\sum_{i=1}^n p_i x_i, \sum_{j=1}^n p_j x_j])) \leq \sum_{i=1}^n p_i r_i^2$$

or

$$p(\sum_{i=1}^{n} p_i[x_i, x_i] - [\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j] - [\sum_{j=1}^{n} p_j x_j, \sum_{i=1}^{n} p_i x_i] + [\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j]) \leq \sum_{i=1}^{n} p_i r_i^2$$

which leads to

$$p(\sum_{i=1}^{n} p_i[x_i, x_i] - [\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j]) \leq \sum_{i=1}^{n} p_i r_i^2$$

By $p(y_1) - p(y_2) \leq p(y_1 - y_2)$, $(\forall)\ y_1,\ y_2 \in Z$ we obtain the second inequality (3).

Let new assume that $p_1,\ p_2 \in (0,\ 1)$ with $p_1 + p_2 = 1$ and $x_1,\ x_2 \in \mathcal{H}$. For $r_1 = p_2 q_p(x_1 - x_2)$ and $r_2 = p_1 q_p(x_1 - x_2)$ the inequalities (1) will be

$$q_p(x_1 - p_1 x_1 - p_2 x_2) \leq r_1$$

and

$$q_p(x_2 - p_1 x_1 - p_2 x_2) \leq r_1.$$

If we assume that there exists the constant $c > 0$ such that

$$p(\sum_{i=1}^{n} p_i[x_i, x_i] - [\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} p_j x_j]) \leq c \sum_{i=1}^{n} p_i r_i^2, \tag{2}$$

to be true, we shall show that $c \geq 1$.

If we compute the left term of the above inequality (2) we have

$$p(p_1[x_1, x_1] + p_2[x_2, x_2] - [p_1 x_1 + p_2 x_2, p_1 x_1 + p_2 x_2]) = p(p_1[x_1, x_1] + p_2[x_2, x_2] -$$
$$- p_1^2[x_1, x_1] - p_2^2[x_2, x_2] - p_1 p_2([x_2, x_1] + [x_1, x_2])) = p(p_1(1-p_1)[x_1, x_1] + p_2(1-p_2)[x_2, x_2] -$$
$$- p_1 p_2([x_2, x_1] + [x_1, x_2])) = p_1 p_2 p([x_1, x_1] + [x_2, x_2] - [x_2, x_1] - [x_1, x_2]) =$$
$$= p_1 p_2 p([x_1 - x_2, x_1 - x_2]) = p_1 p_2 q_p^2(x_1 - x_2).$$

The right member, is

$$p_1 r_1^2 + p_2 r_2^2 = p_1 p_2^2 q_p^2(x_1 - x_2) + p_2 p_1^2 q_p^2(x_1 - x_2) = p_1 p_2 q_p^2(x_1 - x_2).$$

Then the inequality (2) becomes

$$p_1 p_2 q_p^2(x_1 - x_2) \leq c p_1 p_2 q_p^2(x_1 - x_2),$$

that is $c \geq 1$.

□

## References


[1] Chobanyan S. A and Weron A., Banach-space-valued stationary processes and their linear prediction, Dissertations Math., 125, (1975), 1–45.

[2] L. Ciurdariu, Classes of linear operators on Pseudo-Hilbert Spaces and Applications, Part I, Monografii matematice, Tipografia Universitatii de Vest din Timisoara, 2006.

[3] L. Ciurdariu, A Generalization of Bergstrom and Radon′s Inequalities in Pseudo-Hilbert Spaces, Preprint *RGMIA Res. Rep. Coll.*, 12 (2) (2009), Article 18, pp. 1-6, (electronic).

[4] S. S. Dragomir, Some Inequalities for Commutators of Bounded Linear Operators in Hilbert Spaces, Preprint *RGMIA Res. Rep. Coll.*, 11 (1) (2008), Article 7, pp. 1-10, (electronic).

[5] S. S. Dragomir, Y. J. Cho and S. S. Kim, Some inequalities in inner product spaces related to the generalized triangle inequality, Preprint *RGMIA Res. Rep. Coll.*, 12 (3) (2009), Article 13, pp. 1-10, (electronic).

[6] S. S. Dragomir, A. Sofo, On some inequalities of Cauchy-Bunyakovsky-Schwarz type and applications, Preprint *RGMIA Res. Rep. Coll.*, 10 (2) (2007), Article 3, pp. 1-10, (electronic).

[7] Loynes R. M., Linear operators in $VH$-spaces, *Trans. American Math. Soc.*, 116, (1965), 167-180.

[8] Loynes R. M., On generalized positive definite functions, *Proc. London Math. Soc*, 3, (1965), 373-384.

[9] S. Stratila, L. Zsido, Operator algebras, Monografii Matematice, Part I, Part II, Tipografia Universitatii din Timisoara, 53, (1995), 1-511.

[10] Weron A. and Chobanyan S. A., Stochastic processes on pseudo-Hilbert spaces (russian), *Bull. Acad. Polon.*, Ser. Math. Astr. Phys., tom XX1, 9, (1973), 847-854.

[11] Weron A., Prediction theory in Banach spaces, *Proc. of Winter School on Probability*, Karpacz, Springer Verlag, London, 1975, 207-228.

Department of Mathematics,"Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara, ROMANIA

*E-mail address*, L. Ciurdariu: cloredana43@yahoo.com