.

THE TWO-SIDED INEQUALITIES FOR THE EULER-MASCHERONI CONSTANT

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Abstract. Let $\gamma = 0.577215\ldots$ be the Euler-Mascheroni constant, and let $R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right)$. We prove that for all integers $n \ge 1$,

$$\frac{7}{960(n+a)^4} \le \gamma - R_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} < \frac{7}{960(n+b)^4}$$

with the best possible constants

$$a = \frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(\frac{3}{2}\right) + \gamma - \frac{53}{54}\right]}} - 1 = 0.57027\ldots \quad \text{and} \quad b = \frac{1}{2}.$$

This refines the result of D. W. DeTemple, who proved that the two-sided inequalities hold with $a = 1$ and $b = 0$. Also, the monotonicity properties of functions related to the psi function are obtained.

1. Introduction

Euler's constant $\gamma = 0.57721566490153286\ldots$ was first introduced by Leonhard Euler (1707-1783) in 1734 as

$$\gamma = \lim_{n\to\infty} D_n, \quad \text{where} \quad D_n = \sum_{k=1}^{n} \frac{1}{k} - \log n. \tag{1}$$

It is also known as the Euler-Mascheroni constant. According to Glaisher [11], the use of the symbol γ is probably due to the geometer Lorenzo Mascheroni (1750-1800) who used it in 1790 while Euler used the letter C. The constant γ is deeply related to the gamma function $\Gamma(x)$ thanks to the Weierstrass formula

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{p=1}^{\infty} \left(1 + \frac{x}{p}\right) e^{-x/p}.$$

The Euler-Mascheroni constant plays an important role in Analysis (Gamma function, Bessel functions, exponential-integral, ...) and occurs frequently in Number Theory (order of magnitude of arithmetical functions for instance [12]).

Direct use of formula (1) to compute the Euler-Mascheroni constant is of poor interest since the convergence is very slow. The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers $H_n =$

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$\sum_{k=1}^{n} \frac{1}{k}$,

$$H_n - \log n = \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$$

where the B_{2k} are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2}$$

First four Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \tag{3}$$

and then

$$\gamma = H_n - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots .$$

Several bounds for $D_n - \gamma$ have been given in the literature. We recall some of them:

$$\begin{aligned}
\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2(n-1)} \quad &\text{for} \quad n \ge 2 \quad ([18]);\\
\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad &\text{for} \quad n \ge 1 \quad ([15, 21]);\\
\frac{1-\gamma}{n} \le D_n - \gamma < \frac{1}{2n} \quad &\text{for} \quad n \ge 1 \quad ([4]);\\
\frac{1}{2n + \frac{2}{5}} < D_n - \gamma < \frac{1}{2n + \frac{1}{3}} \quad &\text{for} \quad n \ge 1 \quad ([19, 20]);\\
\frac{1}{2n + \frac{2\gamma-1}{1-\gamma}} \le D_n - \gamma < \frac{1}{2n + \frac{1}{3}} \quad &\text{for} \quad n \ge 1 \quad ([3, 6, 19, 20]).
\end{aligned}$$

See also [16, 17].

The convergence of the sequence D_n to γ is very slow. In 1993, D. W. DeTemple [9] studied a modified sequence which converges faster and proved

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, \tag{4}$$

$$\frac{7}{960(n+1)^4} < \gamma - R_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} < \frac{7}{960n^4}, \tag{5}$$

where

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \log\left(n + \frac{1}{2}\right).$$

Now let

$$H(n) = n^2(R_n - \gamma), \quad n \ge 1.$$

Since

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k},$$

where $\psi = \Gamma'/\Gamma$ is the psi function, we see that

$$H(n) = (R_n - \gamma)n^2 = \left[\psi(n+1) - \log\left(n + \frac{1}{2}\right)\right] n^2.$$

Some computer experiments led M. Vuorinen to conjecture that $H(n)$ increases on the interval $[1,\infty)$ from $H(1)=-\gamma+1-\log(3/2)=0.0173\dots$ to $1/24=0.0416\dots$. E. A. Karatsuba [13] proved that for all integers $n\ge 1$, $H(n)<H(n+1)$, by clever use of Stirling formula and Fourier series. Some computer experiments also seem to indicate that $[(n+1)/n]^2H(n)$ is a decreasing convex function [5]. The author [**?**] verified that for all integers $n\ge 1$, $H(n)$ and $[(n+1/2)/n]^2H(n)$ are both strictly increasing concave sequences, while $[(n+1)/n]^2H(n)$ is strictly decreasing log-convex sequence.

By the asymptotic formula [2, p. 550]

$$\psi(x)=\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-1/2)^2}+O(x^{-4})\quad \text{as}\quad x\to\infty, \tag{6}$$

we conclude that

$$\lim_{n\to\infty}H(n)=\lim_{n\to\infty}[(n+1/2)/n]^2H(n)=\lim_{n\to\infty}[(n+1)/n]^2H(n)=\frac{1}{24}. \tag{7}$$

From the increasingness of $H(n)$, decreasingness of $[(n+1)/n]^2H(n)$ and (7), we obtain the inequality (4). From the increasingness of $[(n+1/2)/n]^2H(n)$, decreasingness of $[(n+1)/n]^2H(n)$ and (7), we get that

$$\frac{1}{24(n+1)^2}<R_n-\gamma<\frac{1}{24(n+\frac{1}{2})^2},\quad n\ge 1. \tag{8}$$

Obviously, the upper in (8) is sharper than one in (4). We remark that the second inequality in (8) comes out from what D. W. DeTemple [9] wrote on page 470 of the article. Also, A. Sîntămărian [16] gave results for a generalization of Euler's constant and taken $a=1$ in [16, Theorem 3.1, part (iii)] we obtain the second inequality in (8).

Recently, the author [8] proved that for all integers $n\ge 1$, then

$$\frac{1}{24(n+a)^2}\le R_n-\gamma<\frac{1}{24(n+b)^2} \tag{9}$$

with the best possible constants

$$a=\frac{1}{\sqrt{24[-\gamma+1-\log(3/2)]}}-1=0.55106\dots\quad \text{and}\quad b=\frac{1}{2}.$$

The inequality (5) can be written as

$$\frac{7}{960(n+1)^4}<\log\left(n+\frac{1}{2}\right)-\psi(n+1)+\frac{1}{24\left(n+\frac{1}{2}\right)^2}<\frac{7}{960n^4}. \tag{10}$$

Motivated by the inequality (10), we establish the following results.

Theorem 1. *Let $a\ge 0$ be a real number and $J_a(x)$ be defined by*

$$J_a(x)=(x+a)^4\left[\log\left(x+\frac{1}{2}\right)-\psi(x+1)+\frac{1}{24\left(x+\frac{1}{2}\right)^2}\right]. \tag{11}$$

Then, the functions $J_{1/2}$ on $(-1/2,\infty)$ and J_0 on $(0,\infty)$ are strictly increasing.

Remark 1. By the asymptotic formula [2, p. 550]

$$\psi(x)=\log\left(x-\frac{1}{2}\right)+\frac{1}{24(x-1/2)^2}-\frac{7}{960(x-\frac{1}{2})^4}+O(x^{-6})\quad \text{as}\quad x\to\infty,$$

we conclude that

$$\lim_{x\to\infty} J_0(x) = \lim_{x\to\infty} J_{1/2}(x) = \lim_{x\to\infty} J_1(x) = \frac{7}{960}. \tag{12}$$

From the increasingness of $J_0(x)$, decreasingness of $J_1(x)$ and (12), we obtain the inequality (5). From the increasingness of $J_{1/2}(x)$, decreasingness of $J_1(x)$ and (12), we get that

$$\frac{7}{960(n+1)^4} < \gamma - R_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} < \frac{7}{960(n+\frac{1}{2})^4}, \quad n \geq 1. \tag{13}$$

Obviously, the upper in (13) is sharper than one in (5).

Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{14}$$

for $x \in I$ and $n \geq 0$. Dubourdien [10] pointed out that if a non-constant function f is completely monotonic, then strict inequality holds in (14). Recall that a function f is said to be a Bernstein function on an interval I if $f > 0$ and f' is completely monotonic on I.

By Theorem 1, we pose the following conjecture.

Corollary 1. *Let J_a be defined by (11). Then, the functions $J_{1/2}$ on $(-1/2, \infty)$ and J_0 on $(0, \infty)$ are Bernstein function, while the function J_1 is completely monotonic on $(0, \infty)$.*

In view of the inequality (13) it is natural to ask: What is the smallest number a and what is the largest number b such that the inequality

$$\frac{7}{960(n+a)^4} \leq \gamma - R_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} \leq \frac{7}{960(n+b)^4},$$

holds for all integers $n \geq 1$? The following Theorem 2 answers this question.

Theorem 2. *For all integers $n \geq 1$, then*

$$\frac{7}{960(n+a)^4} \leq \gamma - R_n + \frac{1}{24\left(n+\frac{1}{2}\right)^2} < \frac{7}{960(n+b)^4} \tag{15}$$

with the best possible constants

$$a = \frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(\frac{3}{2}\right) + \gamma - \frac{53}{54}\right]}} - 1 = 0.57027\ldots \quad \textit{and} \quad b = \frac{1}{2}.$$

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Define for $x > 0$,

$$f(x) = x^4 \left[\log x - \psi\left(x + \frac{1}{2}\right) + \frac{1}{24x^2}\right].$$

Using the representations [1, p. 259]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt, \tag{16}$$

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt \tag{17}$$

and

$$\frac{1}{x^2} = \int_0^\infty te^{-xt}dt, \tag{18}$$

we imply

$$f(x) = x^4 \int_0^\infty \mu(t)e^{-xt}dt, \tag{19}$$

where

$$\mu(t) = -\frac{1}{t} + \frac{1}{e^{t/2} - e^{-t/2}} + \frac{1}{24}t, \quad t > 0. \tag{20}$$

Easy computations reveal that

$$\begin{aligned}\frac{f'(x)}{x^3} &= 4\int_0^\infty \mu(t)e^{-xt}dt - x\int_0^\infty t\mu(t)e^{-xt}dt \\ &= 4\int_0^\infty \mu(t)e^{-xt}dt - \int_0^\infty [\mu(t) + t\mu'(t)]e^{-xt}dt \\ &= \int_0^\infty [3\mu(t) - t\mu'(t)]e^{-xt}dt.\end{aligned}$$

It is easy to see that for $t > 0$,

$$\begin{aligned}3\mu(t) - t\mu'(t) > 0 &\iff -\frac{4}{t} + \frac{3}{e^{t/2} - e^{-t/2}} + \frac{t}{12} + \frac{t(e^{t/2} + e^{-t/2})}{2(e^{t/2} - e^{-t/2})^2} > 0 \\ &\iff -\frac{2}{u} + \frac{3}{2\sinh u} + \frac{u}{6} + \frac{u\cosh u}{2(\sinh u)^2} > 0 \quad (\text{where} \quad u = t/2) \\ &\iff \frac{-12(\sinh u)^2 + 9u\sinh u + u^2(\sinh u)^2 + 3u^2\cosh u}{6u(\sinh u)^2} > 0.\end{aligned}$$

Define for $u > 0$,

$$g(u) = -12(\sinh u)^2 + 9u\sinh u + u^2(\sinh u)^2 + 3u^2\cosh u.$$

Then,

$$\begin{aligned}g(u) &= -6[\cosh(2u) - 1] + 9u\sinh u + \frac{u^2[\cosh(2u) - 1]}{2} + 3u^2\cosh u \\ &= \sum_{n=4}^\infty \frac{2n(2n-1)(2^{2n-3} + 3) + 18n - 3\cdot 2^{2n+1}}{(2n)!}u^{2n} > 0, \quad u > 0.\end{aligned}$$

Hence, $f'(x) > 0$ for $x > 0$. Clearly, the function

$$J_{1/2}(x) = \left(x + \frac{1}{2}\right)^4 \left[\log\left(x + \frac{1}{2}\right) - \psi(x+1) + \frac{1}{24\left(x + \frac{1}{2}\right)^2}\right]$$

is strictly increasing on $(-\frac{1}{2}, \infty)$. It is easy to see that the function

$$J_0(x) = \left(\frac{x}{x + \frac{1}{2}}\right)^4 J_{1/2}(x)$$

is strictly increasing on $(0, \infty)$. The proof is complete. □

In order prove our Theorem 2 we need to the following results [2]: For $x > \frac{1}{2}, N = 0, 1, 2, \ldots,$

$$\log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}}<\psi(x) < \log\left(x-\frac{1}{2}\right)-\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} \tag{21}$$

and

$$\frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}}<(-1)^{n+1}\psi^{(n)}(x) < \frac{(n-1)!}{(x-\frac{1}{2})^n}+\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}}, \quad n=1,2,\ldots, \tag{22}$$

where

$$B_k(1/2)=-\left(1-\frac{1}{2^{k-1}}\right)B_k, \quad k=0,1,2,\ldots,$$

B_k are Bernoulli numbers defined by (2). By (3) we get

$$B_2(1/2)=-\frac{1}{12}, \quad B_4(1/2)=\frac{7}{240}, \quad B_6(1/2)=-\frac{31}{1344}, \quad B_8(1/2)=\frac{127}{3840}.$$

From (21), we obtain for $x > \frac{1}{2}$,

$$\psi(x)-\log\left(x-\frac{1}{2}\right)<\frac{1}{24(x-\frac{1}{2})^2}-\frac{7}{960(x-\frac{1}{2})^4}+\frac{31}{8064(x-\frac{1}{2})^6}. \tag{23}$$

From (22), we obtain for $x > \frac{1}{2}$,

$$\frac{1}{12(x-\frac{1}{2})^3}-\frac{7}{240(x-\frac{1}{2})^5}+\frac{31}{1344(x-\frac{1}{2})^7}-\frac{127}{3840(x-\frac{1}{2})^9}<\frac{1}{x-\frac{1}{2}}-\psi'(x). \tag{24}$$

Now we are in position to prove our Theorem 2.

Proof of Theorem 2. The inequality (15) can be written as

$$a \geq \frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(n+\frac{1}{2}\right)-\psi(n+1)+\frac{1}{24\left(n+\frac{1}{2}\right)^2}\right]}}-n>b.$$

In order to prove (15) we define

$$f(x)=\frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(x+\frac{1}{2}\right)-\psi(x+1)+\frac{1}{24\left(x+\frac{1}{2}\right)^2}\right]}}-x.$$

Differentiation yields

$$\begin{aligned}
&\left[\frac{960}{7}\left(\log\left(x+\frac{1}{2}\right)-\psi(x+1)+\frac{1}{24(x+\frac{1}{2})^2}\right)\right]^{5/4} f'(x)\\
&=\frac{240}{7}\left(-\frac{1}{x+\frac{1}{2}}-\psi'(x+1)+\frac{1}{12(x+\frac{1}{2})^3}\right)\\
&\qquad\qquad -\left[\frac{960}{7}\left(\log\left(x+\frac{1}{2}\right)-\psi(x+1)+\frac{1}{24(x+\frac{1}{2})^2}\right)\right]^{5/4}\\
&<\frac{240}{7}\left[\frac{7}{240(x+\frac{1}{2})^5}-\frac{31}{1344(x+\frac{1}{2})^7}+\frac{127}{3840(x+\frac{1}{2})^9}\right]\\
&\qquad\qquad -\left[\frac{960}{7}\left(\frac{7}{960(x+\frac{1}{2})^4}-\frac{31}{8064(x+\frac{1}{2})^6}\right)\right]^{5/4}\\
&=\frac{1}{u^5}\left[1-\frac{155}{196u^2}+\frac{127}{112u^4}-\left(1-\frac{155}{294u^2}\right)^{5/4}\right],
\end{aligned}$$

where $u = x + \frac{1}{2}$.

Now we show that there exists a positive real number x_0 such that $f'(x_0) < 0$ for $x > x_0$. In order to find x_0, we consider

$$1-\frac{155}{196u^2}+\frac{127}{112u^4}<\left(1-\frac{155}{294u^2}\right)^{5/4}. \tag{25}$$

By Bernoulli's inequality: Let $x \geq -1$, then for $\alpha < 0$ or $\alpha > 1$, $(1+x)^\alpha \geq 1+\alpha x$, the equal sign holds if and only if $x = 0$, we have

$$1-\frac{775}{1176u^2}<\left(1-\frac{155}{294u^2}\right)^{5/4}, \quad u > 0.726\ldots. \tag{26}$$

The inequality

$$1-\frac{155}{196u^2}+\frac{127}{112u^4}<1-\frac{775}{1176u^2} \tag{27}$$

holds for $u > 31.041\ldots$, and then, $f'(x) < 0$ for $x > 30.541\ldots$. Straightforward calculation produces

$$\begin{aligned}
&f(1)=0.57027\ldots, f(2)=0.54774\ldots, f(3)=0.53564\ldots, f(4)=0.52830\ldots,\\
&f(5)=0.52341\ldots, f(6)=0.52268\ldots, f(7)=0.51725\ldots, f(8)=0.51519\ldots,\\
&f(9)=0.51376\ldots, f(10)=0.51246\ldots, f(11)=0.51139\ldots, f(12)=0.51049\ldots,\\
&f(13)=0.50972\ldots, f(14)=0.50905\ldots, f(15)=0.50847\ldots, f(16)=0.50794\ldots,\\
&f(17)=0.50751\ldots, f(18)=0.50705\ldots, f(19)=0.50674\ldots, f(20)=0.50615\ldots,\\
&f(21)=0.50593\ldots, f(22)=0.50584\ldots, f(23)=0.50559\ldots, f(24)=0.50537\ldots,\\
&f(25)=0.50513\ldots, f(26)=0.50496\ldots, f(27)=0.50476\ldots, f(28)=0.50459\ldots,\\
&f(29)=0.50444\ldots, f(30)=0.50431\ldots, f(31)=0.50417\ldots.
\end{aligned}$$

Thus, the sequence

$$f(n)=\frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(n+\frac{1}{2}\right)-\psi(n+1)+\frac{1}{24\left(n+\frac{1}{2}\right)^2}\right]}}-n \quad (n=1,2,\ldots)$$

is strictly decreasing. This leads to

$$\lim_{n\to\infty} f(n) < f(n) \le f(1) = \frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(\frac{3}{2}\right) + \gamma - \frac{53}{54}\right]}} - 1 = 0.57027\ldots. \tag{28}$$

It remains to prove that

$$\lim_{n\to\infty} f(n) = \frac{1}{2}. \tag{29}$$

From the asymptotic formula [2, p. 550]

$$\psi(x) = \log\left(x - \frac{1}{2}\right) + \frac{1}{24(x-\frac{1}{2})^2} - \frac{7}{960(x-\frac{1}{2})^4} + \frac{31}{8064(x-\frac{1}{2})^6} + O(x^{-8})$$

as $x \to \infty$, we obtain

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt[4]{\frac{960}{7}\left[\log\left(x+\frac{1}{2}\right) - \psi(x+1) + \frac{1}{24\left(x+\frac{1}{2}\right)^2}\right]}} - x \\
&= \frac{1 - x\sqrt[4]{\frac{960}{7}\left[\log\left(x+\frac{1}{2}\right) - \psi(x+1) + \frac{1}{24\left(x+\frac{1}{2}\right)^2}\right]}}{\sqrt[4]{\frac{960}{7}\left[\log\left(x+\frac{1}{2}\right) - \psi(x+1) + \frac{1}{24\left(x+\frac{1}{2}\right)^2}\right]}} \\
&= \frac{1 - x\sqrt[4]{\frac{1}{(x+1/2)^4} - \frac{155}{294(x+1/2)^6} + O(x^{-8})}}{\sqrt[4]{\frac{1}{(x+1/2)^4} - \frac{155}{294(x+1/2)^6} + O(x^{-8})}} \\
&= \frac{x + \frac{1}{2} - x\sqrt[4]{1 - \frac{155}{294(x+1/2)^2} + O(x^{-4})}}{\sqrt[4]{1 - \frac{155}{294(x+1/2)^2} + O(x^{-4})}} \\
&= \frac{\frac{1}{2} + O(x^{-1})}{1 + O(x^{-2})} \to \frac{1}{2} \quad \text{as} \quad x \to \infty,
\end{aligned}$$

and then,

$$\lim_{n\to\infty} f(n) = \frac{1}{2}. \tag{30}$$

The proof is complete. □

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