FEJÉR-TYPE INEQUALITIES (I)

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Abstract. In this paper, we establish some new Fejér-type inequalities for convex functions.

1. Introduction

Throughout this paper, let $f : [a, b] \to \mathbb{R}$ be convex, and $g : [a, b] \to [0, \infty)$ be integrable and symmetric to $\frac{a+b}{2}$. We define the following functions on $[0, 1]$ that are associated with the well known Hermite-Hadamard inequality [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

namely

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right)\right] g(x)\,dx;$$

$$J(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{3a+b}{4}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+3b}{4}\right)\right] g(x)\,dx;$$

$$M(t) = \int_a^{\frac{a+b}{2}} \frac{1}{2}\left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(t\frac{a+b}{2} + (1-t)\frac{x+b}{2}\right)\right] g(x)\,dx + \int_{\frac{a+b}{2}}^b \frac{1}{2}\left[f\left(t\frac{a+b}{2} + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right)\right] g(x)\,dx;$$

and

$$N(t) = \int_a^b \frac{1}{2}\left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right)\right] g(x)\,dx.$$

For some results which generalize, improve, and extend the famous integral inequality (1.1), see [2] – [6].

In [2], Dragomir established the following theorem which is a refinement of the first inequality of (1.1):

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Theorem A. *Let f be defined as above, and let H be defined on* $[0,1]$ *by*

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

Then H is convex, increasing on $[0,1]$, *and for all* $t\in[0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right)=H(0)\le H(t)\le H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

In [6], Yang and Hong established the following theorem which is a refinement of the second inequality in (1.1):

Theorem B. *Let f be defined as above, and let P be defined on* $[0,1]$ *by*

$$P(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right)\right]dx.$$

Then P is convex, increasing on $[0,1]$, *and for all* $t\in[0,1]$, *we have*

$$\frac{1}{b-a}\int_a^b f(x)\,dx=P(0)\le P(t)\le P(1)=\frac{f(a)+f(b)}{2}. \tag{1.3}$$

In [3], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem C. *Let f,g be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \tag{1.4}$$

is known as Fejér inequality.

In this paper, we establish some Fejér-type inequalities related to the functions I,J,M,N introduced above.

2. Main Results

In order to prove our main results, we need the following lemma:

Lemma 1 (see [4])**.** *Let f be defined as above and let $a\le A\le C\le D\le B\le b$ with $A+B=C+D$. Then*

$$f(C)+f(D)\le f(A)+f(B).$$

Now, we are ready to state and prove our results.

Theorem 2. *Let f,g,I be defined as above. Then I is convex, increasing on* $[0,1]$, *and for all* $t\in[0,1]$, *we have the following Fejér-type inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx=I(0)\le I(t)\le I(1)=\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx. \tag{2.1}$$

Proof. It is easily observed from the convexity of f that I is convex on $[0,1]$. Using simple integration techniques and under the hypothesis of g, the following identity holds on $[0,1]$,

$$\begin{aligned} I(t) = \int_a^{\frac{a+b}{2}} \Big[& f\left(tx + (1-t)\frac{a+b}{2}\right) \\ & + f\left(t(a+b-x) + (1-t)\frac{a+b}{2}\right)\Big] g(2x-a)\,dx. \end{aligned} \tag{2.2}$$

Let $t_1 < t_2$ in $[0,1]$. By Lemma 1, the following inequality holds for all $x \in \left[a, \frac{a+b}{2}\right]$:

$$\begin{aligned} & f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + f\left(t_1(a+b-x) + (1-t_1)\frac{a+b}{2}\right) \\ & \le f\left(t_2 x + (1-t_2)\frac{a+b}{2}\right) + f\left(t_2(a+b-x) + (1-t_2)\frac{a+b}{2}\right). \end{aligned} \tag{2.3}$$

Indeed, it holds when we make the choice:

$$\begin{aligned} A &= t_2 x + (1-t_2)\frac{a+b}{2}, \\ C &= t_1 x + (1-t_1)\frac{a+b}{2}, \\ D &= t_1(a+b-x) + (1-t_1)\frac{a+b}{2} \end{aligned}$$

and

$$B = t_2(a+b-x) + (1-t_2)\frac{a+b}{2}$$

in Lemma 1.

Multipling the inequality (2.3) by $g(2x-a)$, integrating both sides over x on $\left[a, \frac{a+b}{2}\right]$ and using identity (2.2), we derive $I(t_1) \le I(t_2)$. Thus I is increasing on $[0,1]$ and then the inequality (2.1) holds. This completes the proof. ∎

Remark 3. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 2. Then* $I(t) = H(t)$ $(t \in [0,1])$ *and the inequality* (2.1) *reduces to the inequality* (1.2), *where* H *is defined as in Theorem A.*

Theorem 4. *Let* f, g, J *be defined as above. Then* J *is convex, increasing on* $[0,1]$, *and for all* $t \in [0,1]$, *we have the following Fejér-type inequality*

$$\begin{aligned} \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} \int_a^b g(x)\,dx &= J(0) \le J(t) \le J(1) \\ &= \frac{1}{2}\int_a^b \left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx. \end{aligned} \tag{2.4}$$

Proof. By using a similar method to that from Theorem 2, we can show that J is convex on $[0,1]$, the identity

$$J(t)=\int_a^{\frac{3a+b}{4}}\left[f\left(tx+(1-t)\frac{3a+b}{4}\right)+f\left(t\left(\frac{3a+b}{2}-x\right)+(1-t)\frac{3a+b}{4}\right)+f\left(t\left(x+\frac{b-a}{2}\right)+(1-t)\frac{a+3b}{4}\right)+f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}\right)\right]g(2x-a)\,dx \tag{2.5}$$

holds on $[0,1]$ and the inequalities

$$\begin{aligned} f\left(t_1x+(1-t_1)\frac{3a+b}{4}\right)+f\left(t_1\left(\frac{3a+b}{2}-x\right)+(1-t_1)\frac{3a+b}{4}\right) \\ \leq f\left(t_2x+(1-t_2)\frac{3a+b}{4}\right)+f\left(t_2\left(\frac{3a+b}{2}-x\right)+(1-t_2)\frac{3a+b}{4}\right), \end{aligned} \tag{2.6}$$

$$\begin{aligned} f\left(t_1\left(x+\frac{b-a}{2}\right)+(1-t_1)\frac{a+3b}{4}\right) + f\left(t_1(a+b-x)+(1-t_1)\frac{a+3b}{4}\right) \\ \leq f\left(t_2\left(x+\frac{b-a}{2}\right)+(1-t_2)\frac{a+3b}{4}\right) + f\left(t_2(a+b-x)+(1-t_2)\frac{a+3b}{4}\right) \end{aligned} \tag{2.7}$$

hold for all $t_1<t_2$ in $[0,1]$ and $x\in\left[a,\frac{3a+b}{4}\right]$.

By (2.5) – (2.7) and using a similar method to that from Theorem 2, we can show that J is increasing on $[0,1]$ and (2.4) holds. This completes the proof. ∎

The following result provides a comparison between the functions I and J.

Theorem 5. *Let f,g,I,J be defined as above. Then $I(t)\leq J(t)$ on $[0,1]$.*

Proof. By the identity

$$J(t)=\int_a^{\frac{a+b}{2}}\left[f\left(tx+(1-t)\frac{3a+b}{4}\right)+f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}\right)\right]g(2x-a)\,dx \tag{2.8}$$

on $[0,1]$, (2.2) and using a similar method to that from Theorem 2, we can show that $I(t)\leq J(t)$ on $[0,1]$. The details are omited. ∎

Further, the following result incorporates the properties of the function M :

Theorem 6. *Let f, g, M be defined as above. Then M is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have the following Fejér-type inequality*

$$\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right] g(x)\,dx = M(0) \le M(t) \le M(1) = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx. \tag{2.9}$$

Proof. Follows by the identity

$$M(t) = \int_a^{\frac{3a+b}{4}} \left[f(ta+(1-t)x) + f\left(t\frac{a+b}{2}+(1-t)\left(\frac{3a+b}{2}-x\right)\right) + f\left(t\frac{a+b}{2}+(1-t)\left(x+\frac{b-a}{2}\right)\right) + f(tb+(1-t)(a+b-x))\right] \times g(2x-a)\,dx \tag{2.10}$$

on $[0,1]$. The details are left to the interested reader. ∎

We now present a result concerning the properties of the function N :

Theorem 7. *Let f, g, N be defined as above. Then N is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have the following Fejér-type inequality*

$$\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right] g(x)\,dx = N(0) \le N(t) \le N(1) = \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2.11}$$

Proof. By the identity

$$N(t) = \int_a^{\frac{a+b}{2}} \left[f(ta+(1-t)x) + f(tb+(1-t)(a+b-x))\right] g(2x-a)\,dx \tag{2.12}$$

on $[0,1]$ and using a similar method to that for Theorem 2, we can show that N is convex, increasing on $[0,1]$ and (2.11) holds. ∎

Remark 8. *Let $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ in Theorem 7. Then $N(t) = P(t)$ $(t \in [0,1])$ and the inequality (2.11) reduces to (1.3) where P is defined as in Theorem B.*

Theorem 9. *Let f, g, M, N be defined as above. Then $M(t) \le N(t)$ on $[0,1]$.*

Proof. By the identity

$$N(t) = \int_a^{\frac{3a+b}{4}} \left[f(ta+(1-t)x) + f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right)\right) + f(tb+(1-t)(a+b-x)) + f\left(tb+(1-t)\left(x+\frac{b-a}{2}\right)\right)\right] g(2x-a)\,dx \tag{2.13}$$

on $[0,1]$, (2.10) and using a similar method to that for Theorem 2, we can show that $M(t) \le N(t)$ on $[0,1]$. This completes the proof. ∎

The following Fejér-type inequality is a natural consequence of Theorems 2 – 9.

Corollary 10. *Let f, g be defined as above. Then we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx \tag{2.14}$$
$$\le \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx$$
$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx$$
$$\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.$$

Remark 11. *Let $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ in Corollary 10. Then the inequality* (2.14) *reduces to*

$$f\left(\frac{a+b}{2}\right) \le \frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2} \le \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \le \frac{f(a)+f(b)}{2},$$

which is a refinement of (1.1)*.*

Remark 12. *In Corollary 10, the third inequality in* (2.14) *is the weighted generalization of Bullen's inequality* [5]

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

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