The Euler's number and some means

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Abstract

We investigate several families of means interpolating between the geometric and arithmetic meas and find the one providing the best convergence of

$$\left(1+\frac{1}{n}\right)^{\mathcal{M}_t(n+1,n)}$$

to Euler's number.

1 Introduction

The following problem was proposed by Mihaly Bencze in the Octagon magazine [1]:

$$e < \left(1+\frac{1}{n}\right)^{\left(\frac{(n)^{1/3}+(n+1)^{1/3}}{2}\right)^3}. \tag{1}$$

Although the solution to the problem is known for a long time, it is interesting to generalize it.
Let us use standard notation:

$$G(x,y)=\sqrt{xy}, \quad L(x,y)=\frac{x-y}{\log x-\log y}, \quad A(x,y)=\frac{x+y}{2}$$

for the geometric, logarithmic and arithmetic means of positive numbers x, y respectively. The inequalities

$$G(x,y) < L(x,y) < A(x,y) \tag{2}$$

hold for all $x \neq y$ (see [3] and references therein).
The logarithmic mean is linked with the Euler number by a simple and elegant formula

$$e = \left(1+\frac{1}{n}\right)^{L(n+1,n)}, \tag{3}$$

This, together with the basic property of means - lying in between - leads to the sequence of inequailities

$$\left(1+\frac{1}{n}\right)^{n} < \left(1+\frac{1}{n}\right)^{G(n+1,n)} < e < \left(1+\frac{1}{n}\right)^{A(n+1,n)} < \left(1+\frac{1}{n}\right)^{n+1}$$

The folowing question arises: suppose we have a continuous interpolation between the geometric and arithmetic means, i.e. a family of means

$$\mathcal{M}_t(x,y),\ 0 \le t \le 1, \quad \mathcal{M}_0(x,y) = G(x,y) \text{ and } \mathcal{M}_1(x,y) = A(x,y)$$

that is monotone in t. For which value of t the sequence

$$\left(1+\frac{1}{n}\right)^{\mathcal{M}_t(n+1,n)}$$

assures the fastest convergence to e?
The aim of this paper is to answer this question in case of some known families of means.

Considering (3) our task reduces to finding t such that $\mathcal{M}_t(n+1,n)$ is closest to $L(n+1,n)$. Since all means we consider are homogeneous in x, y, it is enough to investigate the behaviour of $L(x,1) - \mathcal{M}_t(x,1)$ for $x \approx 1$, or that of $L(e^t,1) - \mathcal{M}_t(e^t,1)$ in a neighborhood of 0.

2 Generalized Heronian means

Probably the most natural interpolation between A and G is their convex combination ([4]) given by

$$\mathrm{He}_p(x,y) = (1-p)G(x,y) + pA(x,y), \quad 0 \le p \le 1,$$

named after Hero of Alexandria, who used $\mathrm{He}_{2/3} = \frac{x+\sqrt{xy}+y}{3}$ to calculate the volume of a frustum of a piramid. The global inequality $L < \mathrm{He}_{1/3}$ was obtained first by Carlsson ([3]) and reproved by Janous ([4]). In particular for all $p \ge 1/3$ the inequality

$$L(n+1,n) < \mathrm{He}_p(n+1,n) \tag{4}$$

holds.

Theorem 1. *If $p \ge 1/3$, then (4) holds for all natural n. If $0 < p < 1/3$, then there exists $N = N(p)$ such that reversed (4) is valid for $n > N$.*

Proof.

$$\mathrm{He}_p(e^{2t},1) - L(e^{2t},1) = (1-p)e^t + \frac{e^{2t}+1}{2} - \frac{e^{2t}-1}{2t} \tag{5}$$

$$= e^t\left(1-p+p\cosh t - \frac{\sinh t}{t}\right) = e^t \sum_{n=1}^{\infty}\left(p - \frac{1}{2n+1}\right)\frac{t^{2n}}{(2n)!} \tag{6}$$

Clearly for $p \ge 1/3$ all the coefficients are positive and so is the left-hand side which proofs (5), while for $0 < p < 1/3$ the difference is concave at $t = 0$, hence negative for small t and tends to infinity with growing t. Note that we obtain (4) or its reverse by setting $t = \log\sqrt{\frac{n+1}{n}}$. □

3 Hölder means

The Hölder mean [6], also known as power mean [2, 6] or generalized mean [7, 6], of order p is given as

$$M_p(x,y) = \begin{cases} \max(x,y) & p=\infty, \\ \left(\dfrac{x^p+y^p}{2}\right)^{1/p} & p\neq 0, \\ G(x,y) & p=0, \\ \min(x,y) & p=-\infty. \end{cases} \tag{7}$$

With this definition and taking (3) into account, the original problem (1) can be rewritten as
Show that if $p = 1/3$, then for all natural n

$$L(n+1,n) \leq M_p(n+1,n). \tag{8}$$

We shall prove the following

Theorem 2. *If $p \geq 1/3$, then (8) holds for all natural n. If $p \leq 0$, then for all natural n reverse inequality holds. If $0 < p < 1/3$, then there exists $N = N(p)$ such that reversed (8) is valid for $n > N$.*

Proof. The second part follows from inequality $M_0 = G < L$.
Tung-Po Lin proved in [5] that for all $x \neq y$ the inequality

$$L(x,y) < M_{1/3}(x,y)$$

holds, which combined with monotonicity of power means completes the first part of the proof. For $0 < p < 1/3$ L and M_p are not comparable, but in a neighborhood of $x = 1$ they behave nicely. Consider the function

$$f_p(x) = \frac{p\ln x}{2^p} + \frac{(1-x^p)}{(1+x)^p}$$

By simple computation we see that $f_p(1) = f_p'(1) = f_p''(1) = 0$ and $f_p'''(1) = -\frac{p^2}{2^{p+2}}(p-3)$ hence for $0 < p < 1/3$ there is a $\delta > 1$ such that $f_{1/p}(x) < 0$ if $x \in (1, \delta)$. Therefore, for sufficiently large n

$$f_{1/p}\left(\left(1+\frac{1}{n}\right)^p\right) < 0,$$

which is equivalent to $L(n+1,n) > M_p(n+1,n)$. □

4 Heinz means

Heinz means, defined by

$$H_\alpha(x,y) = \frac{x^\alpha y^{1-\alpha} + x^{1-\alpha}y^\alpha}{2}, \quad 0 \leq \alpha \leq 1/2. \tag{9}$$

We have $H_0 = A$ and $H_{1/2} = G$, moreover they decrease with α. As above, we are asking if the inequality

$$\left(1+\frac{1}{n}\right)^{H_\alpha(n+1,n)} < e \tag{10}$$

or its reverse holds.
Comparison between Heinz means and the logarithmic mean was investigated Pittenger in [8]. The proof below just differs from the original one in details.

Theorem 3. *If $\alpha \geq \frac{3-\sqrt{3}}{6}$, then (10) holds for all n. If $\frac{3-\sqrt{3}}{6} < \alpha < 1$, then there exists $N = N(\alpha)$ such that reversed (10) holds for $n > N$.*

Proof. We shall show a stronger fact, that for $\alpha \geq \frac{3-\sqrt{3}}{6}$ the logarithmic mean is always greater than the Heinz mean. Let $\beta = 1/2 - \alpha$. As in case of Heronian means we let $y = 1$ and $x = e^{2t}$. We have

$$\begin{aligned} L(e^{2t},1) - H_\alpha(e^{2t},1) &= \frac{e^{2t}-1}{2t} - \frac{e^{2t(1/2+\beta)} + e^{2t(1/2-\beta)}}{2} \\ &= e^t\left[\frac{\sinh t}{t} - \cosh 2\beta t\right] = \sum_{n=0}^{\infty} a_n(\beta)\frac{t^{2n}}{(2n)!} \end{aligned} \tag{11}$$

where

$$a_n(\beta) = \frac{1}{2n+1} - (2\beta)^{2n} \tag{12}$$

For $\beta \leq \sqrt{3}/6$ we have $a_0(\beta) = 0$, $a_1(\beta) \geq a_1(\sqrt{3}/6) = 0$ and $a_n(\beta) > 0$ for $n \geq 2$, which yields $L > H_\alpha$. To prove the second part observe, that for $\beta > \sqrt{3}/6$ $a_0 = 0$ and $a_1(\beta) < 0$, which means that the left-hand side of (11) is negative in a neighborhood of 0. For n large enough $\log\sqrt{\frac{n+1}{n}}$ falls in this neighborhood, and (11) becomes equivalent to reversed (10). □

5 Geometric interpolation

In section 2 we apply linear interpolation between A and G. In this section we deal with the family

$$G_\alpha(x,y) = G^{1-\alpha}(x,y)A^\alpha(x,y) \quad 0 \leq \alpha \leq 1.$$

Theorem 4. *If $\alpha \leq 1/3$, then for all $x \neq y$ $G_\alpha(x,y) < L(x,y)$.
If $\alpha > 1/3$, then there exists $N = N(\alpha)$ such that for $n > N$ $G_\alpha(n+1,n) > L(n+1,n)$.*

Proof. As above setting $y = 1, x = e^{2t}$ reduces out task to comparison between two fundtions: $g_\alpha(t) = \cosh^\alpha t$ and $l(t) = t^{-1}\sinh t$. We have

$$\lim_{t\to 0^+} l(t)/g_\alpha(t) = 1 \quad \text{and} \quad \lim_{t\to\infty} l(t)/g_\alpha(t) = \infty. \tag{13}$$

Consider the function $h_\alpha(t) = \frac{\sinh t}{\cosh^\alpha t}$. Its second derivative equals

$$h''_\alpha(t) = (1-\alpha)^2 \sinh t \cosh^{-2-\alpha} t \left[\cosh^2 t - 1 - \frac{3\alpha - 1}{(1-\alpha)^2}\right]$$

and we see, that for $\alpha \le 1/3$ h_α is convex, hence its divided difference $t^{-1}h_\alpha(t) = l(t)/g_\alpha(t)$ increases, which, together with (13) completes the first part of our proof.
In case $\alpha > 1/3 \quad h_\alpha$ is concave in some interval $(0, \delta)$ and its divided difference decreases, so we have $l(t) < g_\alpha(t)$ which yields the second part. □

6 Geometric version of Heinz means

The Heinz means are nothing but the arithmetic mean applied to variables interpolating geometrically $\sqrt{xy}$ and $\frac{x+y}{2}$. Here we do a similar construction reversing the roles of the means:

$$K_p(x,y) = G\left(\frac{x+y}{2} + (1-p)\frac{x-y}{2}, \frac{x+y}{2} - (1-p)\frac{x-y}{2}\right), \quad 0 \le p \le 1$$

Theorem 5. *For $p \ge 1 - \sqrt{2/3}$ the inequality $K_p(x,y) > L(x,y)$ holds for all $x \ne y$. If $p < 1 - \sqrt{2/3}$, then there exists $N = N(p)$ such that $K_p(n+1,n) < L(n+1,n)$ is valid for $n > N$.*

The proof is basically the same as in case of Heronian means and we leave it to the reader.

It is interesting to see if the optimal inequalities really makes sense, i.e. wheter the approximations are significantly better that the ones given by the geometric and arithmetic means. To see that good approximation really makes the difference, consider the table below, which shows the values of

$$\log_{10}\left|\left(1+\frac{1}{n}\right)^{\mathcal{M}_p(n+1,n)} - e\right| :$$

n	A	G	$\mathrm{He}_{1/3}$	$M_{1/3}$	$H_{\frac{3-\sqrt{3}}{6}}$	$G_{1/3}$	$K_{\frac{1}{3+\sqrt{6}}}$
10^1	-2.68	-2.98	-7.46	-7.11	-6.51	-7.29	-2.70
10^2	-4.65	-4.95	-11.39	-11.04	-10.43	-11.22	-4.67
10^3	-6.65	-6.95	-15.39	-15.03	-14.43	-15.21	-6.67
10^4	-8.65	-8.95	-19.39	-19.04	-18.44	-19.22	-8.67
10^5	-10.65	-10.95	-23.40	-23.04	-22.44	-23.22	-10.67
10^6	-12.65	-12.95	-27.40	-27.05	-26.45	-27.23	-12.67
10^7	-14.66	-14.96	-31.40	-31.05	-30.45	-31.23	-14.68
10^8	-16.66	-16.96	-35.41	-35.06	-34.46	-35.23	-16.68
10^9	-18.66	-18.96	-39.41	-39.06	-38.46	-39.24	-18.68
10^{10}	-20.66	-20.96	-43.42	-43.06	-42.46	-43.24	-20.68
10^{11}	-22.66	-22.96	-47.43	-47.07	-46.47	-47.25	-22.69
10^{12}	-24.66	-24.97	-51.43	-51.07	-50.47	-51.25	-24.69
10^{13}	-26.67	-26.97	-55.43	-55.08	-54.48	-55.26	-26.69
10^{14}	-28.67	-28.97	-59.44	-59.09	-58.48	-59.26	-28.69
10^{15}	-30.67	-30.97	-63.44	-63.09	-62.49	-63.26	-30.70
10^{16}	-32.67	-32.98	-67.45	-67.10	-66.49	-67.27	-32.70
10^{17}	-34.68	-34.98	-71.45	-71.10	-70.50	-71.27	-34.70
10^{18}	-36.68	-36.98	-75.46	-75.10	-74.50	-75.28	-36.70
10^{19}	-38.68	-38.98	-79.46	-79.11	-78.50	-79.28	-38.70

7 Record breaker

We finish this story with a mean found trough some numerical experiments by the first author.

Theorem 6. *Let*

$$R(x,y) = \frac{14A(x,y) - H(x,y) + 32G(x,y)}{45}.$$

For ever $x \neq y$ the inequality $L(x,y) < R(x,y)$ holds.

Proof.

$$R(e^t,1) - L(e^t,1) = \frac{1}{e^t+1}\sum_{k=1}^{\infty} a_k \frac{t^k}{(2k)!},$$

where $a_k = 32\left(\frac{3^k+1}{2^k}\right) + 12 + \left(7 - \frac{90}{k+1}\right)2^k$ and one can easily calculate that $a_1 = \ldots = a_5 = 0$, $a_6, \ldots, a_{12} > 0$ and for $k > 12$ all a_k's are obviously positive. □

The table below shows why we call it a record breaker:

n	R	n	R	n	R
10^1	-10.18	10^7	-46.09	10^{13}	-82.13
10^2	-16.07	10^8	-52.10	10^{14}	-88.14
10^3	-22.06	10^9	-58.10	10^{15}	-94.14
10^4	-28.07	10^{10}	-64.11	10^{16}	-100.15
10^5	-34.08	10^{11}	-70.12	10^{17}	-106.16
10^6	-40.08	10^{12}	-76.13	10^{18}	-112.16

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