

# NEW INEQUALITIES OF SIMPSON'S TYPE FOR $s$ -CONVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In terms of the first derivative, some inequalities of Simpson's type based on  $s$ -convexity and concavity are introduced. Best Midpoint type inequalities are given. Error estimates for special means and some numerical quadrature rules are also obtained.

## 1. INTRODUCTION

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a four times continuously differentiable mapping on  $(a, b)$  and  $\|f^{(4)}\|_{\infty} := \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$ . The following inequality

$$(1.1) \quad \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4$$

holds, and it is well known in the literature as Simpson's inequality.

It is well known that if the mapping  $f$  is neither four times differentiable nor its fourth derivative  $f^{(4)}$  bounded on  $(a, b)$ , then we cannot apply the classical Simpson quadrature formula.

In recent years many authors have established error estimations for the Simpson's inequality; for refinements, counterparts, generalizations and new Simpson-type inequalities, see [3] – [10], [12] and [19] – [24].

Dragomir in [8] pointed out some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth. Some of the important results are presented below.

**Theorem 1.** *Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable mapping whose derivative is continuous on  $(a, b)$  and  $f' \in L[a, b]$ . Then the following inequality*

$$(1.2) \quad \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)}{3} \|f'\|_1$$

holds, where  $\|f'\|_1 = \int_a^b |f'(x)| dx$ .

The bound of (1.2) for  $L$ -Lipschitzian mappings was given in [8] by  $\frac{5}{36}L(b-a)$ . Also, the following inequality was obtained in [8].

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**Theorem 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $[a, b]$  whose derivative belongs to  $L_p[a, b]$ . Then we have the inequality:

$$(1.3) \quad \left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{6} \left[ \frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p,$$

where,  $(\frac{1}{p}) + (\frac{1}{q}) = 1$ ,  $p > 1$ .

In [15] some inequalities of Hermite-Hadamard type for differentiable convex mappings were presented as follows:

**Theorem 3.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:

$$(1.4) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

A more general result related to (1.4) was established in [16] – [18].

In [8], Hudzik and Maligranda considered among others the class of functions which are  $s$ -convex in the second sense. This class is defined in the following way: a function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ , where  $\mathbb{R}^+ = [0, \infty)$ , is said to be  $s$ -convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all  $x, y \in [0, \infty)$ ,  $\alpha, \beta \geq 0$  with  $\alpha + \beta = 1$  and for some fixed  $s \in (0, 1]$ . This class of  $s$ -convex functions is usually denoted by  $K_s^2$ . It can be easily seen that for  $s = 1$ ,  $s$ -convexity reduces to the ordinary convexity of functions defined on  $[0, \infty)$ .

In [13], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for  $s$ -convex functions in the second sense:

**Theorem 4.** Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is an  $s$ -convex function in the second sense, where  $s \in (0, 1)$  and let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f \in L^1[a, b]$ , then the following inequalities hold:

$$(1.5) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

The constant  $k = \frac{1}{s+1}$  is the best possible in the second inequality in (1.5). The above inequalities are sharp.

For recent results and generalizations concerning Hadamard's inequality see [1, 2] and [14] – [18].

The aim of this paper is to establish Simpson type inequalities based on  $s$ -convexity and concavity. Using these results we can estimate the error( $f$ ) in the Simpson's formula without going through its higher derivatives which may not exist, not be bounded or may be hard to find.

## 2. INEQUALITIES OF SIMPSON TYPE FOR $s$ -CONVEX

In order to prove our main theorems, we need the following lemma:

**Lemma 1.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be an absolutely continuous mapping on  $I^\circ$  where  $a, b \in I$  with  $a < b$ . Then the following equality holds:*

$$(2.1) \quad \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ = (b-a) \int_0^1 p(t) f'(tb + (1-t)a) dt,$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}), \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}.$$

*Proof.* We note that

$$I = \int_0^1 p(t) f'(tb + (1-t)a) dt \\ = \int_0^{1/2} \left(t - \frac{1}{6}\right) f'(tb + (1-t)a) dt + \int_{1/2}^1 \left(t - \frac{5}{6}\right) f'(tb + (1-t)a) dt.$$

Integrating by parts, we get

$$I = \left(t - \frac{1}{6}\right) \frac{f(tb + (1-t)a)}{b-a} \Big|_0^{1/2} - \int_0^{1/2} \frac{f(tb + (1-t)a)}{b-a} dt \\ + \left(t - \frac{5}{6}\right) \frac{f(tb + (1-t)a)}{b-a} \Big|_{1/2}^1 - \int_{1/2}^1 \frac{f(tb + (1-t)a)}{b-a} dt \\ = \frac{1}{6(b-a)} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_0^1 \frac{f(tb + (1-t)a)}{b-a} dt.$$

Setting  $x = tb + (1-t)a$ , and  $dx = (b-a)dt$ , we obtain

$$(b-a) \cdot I = \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dt$$

which gives the desired representation (2.1). ■

The next theorem gives a new refinement of the Simpson inequality for  $s$ -convex functions.

**Theorem 5.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$ , then the following inequality holds:*

$$(2.2) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2} 6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} [|f'(a)| + |f'(b)|].$$

*Proof.* From Lemma 1, and since  $f$  is  $s$ -convex, we have

$$\begin{aligned}
& \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq (b-a) \left| \int_0^1 s(t) f'(tb + (1-t)a) dt \right| \\
& \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\
& \quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt \\
& \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& \quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& = (b-a) \int_0^{1/6} \left( \frac{1}{6} - t \right) (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& \quad + (b-a) \int_{1/6}^{1/2} \left( t - \frac{1}{6} \right) (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& \quad + (b-a) \int_{1/2}^{5/6} \left( \frac{5}{6} - t \right) (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& \quad + (b-a) \int_{5/6}^1 \left( t - \frac{5}{6} \right) (t^s |f'(b)| + (1-t)^s |f'(a)|) dt \\
& = (b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2} 6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} [|f'(a)| + |f'(b)|],
\end{aligned}$$

which completes the proof. ■

Therefore, we can deduce the following result for convex functions.

**Corollary 1.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then the following inequality holds:*

$$\begin{aligned}
(2.3) \quad & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{5(b-a)}{72} [|f'(a)| + |f'(b)|].
\end{aligned}$$

**Remark 1.** *We note that the obtained midpoint inequality (2.3) is better than the inequality (1.2).*

A best upper bound for the midpoint inequality in terms of first derivative may be stated as follows:

**Corollary 2.** *In Theorem 5, if  $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ , then we have,*

$$(2.4) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq (b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2} 6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} [|f'(a)| + |f'(b)|].$$

**Corollary 3.** *In Corollary 2, setting  $s = 1$ , we have,*

$$(2.5) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{5(b-a)}{72} [|f'(a)| + |f'(b)|].$$

**Remark 2.** *We note that the obtained midpoint inequality (2.5) is better than the inequality (1.4).*

The corresponding version of the Simpson's inequality for powers in terms of the first derivative is incorporated in the following result:

**Theorem 6.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^{p/(p-1)}$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $p > 1$ , then the following inequality holds:*

$$(2.6) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}}} \left[ \left( |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} + \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right],$$

where,  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* From Lemma 1, using the well known Hölder integral inequality, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \left| \int_0^1 p(t) f'(tb + (1-t)a) dt \right| \\ & \leq (b-a) \int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| |f'(tb + (1-t)a)| dt \\ & \quad + (b-a) \int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| |f'(tb + (1-t)a)| dt \end{aligned}$$

$$\begin{aligned}
&\leq (b-a) \left( \int_0^{1/2} \left| t - \frac{1}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&\quad + (b-a) \left( \int_{1/2}^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&= (b-a) \left( \int_0^{1/6} \left( \frac{1}{6} - t \right)^p dt + \int_{1/6}^{1/2} \left( t - \frac{1}{6} \right)^p dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&\quad + (b-a) \left( \int_{1/2}^{5/6} \left( \frac{5}{6} - t \right)^p dt + \int_{5/6}^1 \left( t - \frac{5}{6} \right)^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since  $f$  is  $s$ -convex by (1.5), we have

$$(2.7) \quad \int_0^{1/2} |f'(tb + (1-t)a)|^q dt \leq \frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{s+1}$$

and

$$(2.8) \quad \int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \leq \frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{s+1},$$

$$\begin{aligned}
&\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}}} \left[ \left( |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right],
\end{aligned}$$

which completes the proof. ■

**Corollary 4.** Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^{p/(p-1)}$  is convex on  $[a, b]$ , for some fixed  $p > 1$ , then the following inequality holds:

$$\begin{aligned}
(2.9) \quad &\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
&\leq 2^{-\frac{1}{q}} (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \left( |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right],
\end{aligned}$$

where,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Corollary 5.** *In Theorem 6, if in addition  $|f'(a)| = |f'(b)| = 0$ , then*

$$(2.10) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)}{(s+1)^{\frac{1}{q}}} \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left| f'\left(\frac{a+b}{2}\right) \right|,$$

where,  $\frac{1}{p} + \frac{1}{q} = 1$ .

The corresponding version of the midpoint inequality for powers in terms of the first derivative is observed in the following result:

**Corollary 6.** *In Theorem 6, if  $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ , then we have,*

$$(2.11) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \\ \leq (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{1}{(s+1)^{\frac{1}{q}}} \left[ \left( |f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left( \left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right)^{\frac{1}{q}} \right].$$

Another version of the Simpson inequality for powers in terms of the first derivative is obtained as follows:

**Theorem 7.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is  $s$ -convex on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q \geq 1$ , then the following inequality holds:*

$$(2.12) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{(b-a)}{[216(s^2+3s+2)]^{\frac{1}{q}}} \left( \frac{5}{72} \right)^{1-\frac{1}{q}} \left\{ \left[ (3^{-s})(2^{1-s}) + 3s(2^{1-s}) + 3(2^{-s}) \right] |f'(b)|^q \right. \\ \left. + [5^{s+2}3^{-s}2^{1-s} - 6s(2^{-s}) - 21(2^{-s}) + 6s - 24] |f'(a)|^q \right. \\ \left. + \left[ (3^{-s})(2^{1-s}) + 3s(2^{1-s}) + 3(2^{-s}) \right] |f'(a)|^q \right. \\ \left. + [5^{s+2}3^{-s}2^{1-s} - 6s(2^{-s}) - 21(2^{-s}) + 6s - 24] |f'(b)|^q \right\}^{\frac{1}{q}}.$$

*Proof.* Suppose that  $q \geq 1$ . From Lemma 1 and using the well known power mean inequality, we have

$$\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \left| \int_0^1 s(t) f'(tb + (1-t)a) dt \right|$$

$$\begin{aligned}
&\leq (b-a) \int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| |f'(tb + (1-t)a)| dt \\
&\quad + (b-a) \int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| |f'(tb + (1-t)a)| dt \\
&\leq (b-a) \left( \int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| dt \right)^{1-\frac{1}{q}} \left( \int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
&\quad + (b-a) \left( \int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| dt \right)^{1-\frac{1}{q}} \left( \int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since  $|f'|^q$  is  $s$ -convex, therefore we have

$$\begin{aligned}
&\int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| |f'(tb + (1-t)a)|^q dt \\
&\leq \int_0^{1/6} \left( \frac{1}{6} - t \right) (t^s |f'(b)|^q + (1-t)^s |f'(a)|^q) dt \\
&\quad + \int_{1/6}^{1/2} \left( t - \frac{1}{6} \right) (t^s |f'(b)|^q + (1-t)^s |f'(a)|^q) dt \\
&= \frac{(3^{-s})(2^{1-s}) + 3s(2^{1-s}) + 3(2^{-s})}{36(s^2 + 3s + 2)} |f'(b)|^q \\
&\quad + \frac{5^{s+2}3^{-s}2^{1-s} - 6s(2^{-s}) - 21(2^{-s}) + 6s - 24}{36(s^2 + 3s + 2)} |f'(a)|^q
\end{aligned}$$

and

$$\begin{aligned}
&\int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| |f'(tb + (1-t)a)|^q dt \\
&\leq \int_{1/2}^{5/6} \left( \frac{5}{6} - t \right) (t^s |f'(b)|^q + (1-t)^s |f'(a)|^q) dt \\
&\quad + \int_{5/6}^1 \left( t - \frac{5}{6} \right) (t^s |f'(b)|^q + (1-t)^s |f'(a)|^q) dt \\
&= \frac{(3^{-s})(2^{1-s}) + 3s(2^{1-s}) + 3(2^{-s})}{36(s^2 + 3s + 2)} |f'(a)|^q \\
&\quad + \frac{5^{s+2}3^{-s}2^{1-s} - 6s(2^{-s}) - 21(2^{-s}) + 6s - 24}{36(s^2 + 3s + 2)} |f'(b)|^q.
\end{aligned}$$

Also, we note that

$$\int_0^{1/2} \left| \left( t - \frac{1}{6} \right) \right| dt = \int_{1/2}^1 \left| \left( t - \frac{5}{6} \right) \right| dt = \frac{5}{72}.$$

Combining all the above inequalities gives the required result, which completes the proof. ■

**Theorem 8.** Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed



$q \geq 1$ , then the following inequality holds:

$$(2.13) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{72} \left[ \left| f' \left( \frac{29b+61a}{90} \right) \right| + \left| f' \left( \frac{61b+29a}{90} \right) \right| \right].$$

*Proof.* First, we note that by the concavity of  $|f'|^q$  and the power-mean inequality, we have

$$|f'(\alpha x + (1-\alpha)y)|^q \geq \alpha |f'(x)|^q + (1-\alpha) |f'(y)|^q.$$

Hence,

$$|f'(\alpha x + (1-\alpha)y)| \geq \alpha |f'(x)| + (1-\alpha) |f'(y)|,$$

so  $|f'|$  is also concave.

Accordingly, by Lemma 1 and the Jensen integral inequality, we have

$$(2.14) \quad \int_0^{1/2} \left| t - \frac{1}{6} \right| f'(tb + (1-t)a) dt \\ \leq \left( \int_0^{1/2} \left| t - \frac{1}{6} \right| dt \right) \left| f' \left( \frac{\int_0^{1/2} |t - \frac{1}{6}| [tb + (1-t)a] dt}{\int_0^{1/2} |t - \frac{1}{6}| dt} \right) \right| \\ = \frac{5}{72} \left| f' \left( \frac{29b+61a}{90} \right) \right|$$

and

$$(2.15) \quad \int_{1/2}^1 \left| t - \frac{5}{6} \right| f'(tb + (1-t)a) dt \\ \leq \left( \int_{1/2}^1 \left| t - \frac{5}{6} \right| dt \right) \left| f' \left( \frac{\int_{1/2}^1 |t - \frac{5}{6}| [tb + (1-t)a] dt}{\int_{1/2}^1 |t - \frac{5}{6}| dt} \right) \right| \\ = \frac{5}{72} \left| f' \left( \frac{61b+29a}{90} \right) \right|.$$

Therefore,

$$\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{5(b-a)}{72} \left[ \left| f' \left( \frac{29b+61a}{90} \right) \right| + \left| f' \left( \frac{61b+29a}{90} \right) \right| \right],$$

which completes the proof. ■

**Theorem 9.** Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q > 1$ , then the following inequality holds:

$$(2.16) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \left[ \left| f' \left( \frac{3b+a}{4} \right) \right| + \left| f' \left( \frac{b+3a}{4} \right) \right| \right].$$

*Proof.* From Lemma 1, we have

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\ & \quad + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt. \end{aligned}$$

Using the Hölder inequality, for  $q > 1$  and  $p = \frac{q}{q-1}$ , we obtain

$$\begin{aligned} & (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t)a)| dt \\ & \leq (b-a) \left( \int_0^{1/2} \left| t - \frac{1}{6} \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left( \int_0^{1/2} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} & (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t)a)| dt \\ & \leq (b-a) \left( \int_{1/2}^1 \left| t - \frac{5}{6} \right|^{\frac{q}{q-1}} dt \right)^{\frac{q-1}{q}} \left( \int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

It is easy to check that

$$\int_0^{1/2} \left| t - \frac{1}{6} \right|^{\frac{q}{q-1}} dt = \int_{1/2}^1 \left| t - \frac{5}{6} \right|^{\frac{q}{q-1}} dt = \frac{1}{6^{\frac{2q-1}{q-1}}} \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right).$$

Since  $|f'|^q$  is concave on  $[a, b]$  we can use Jensen's integral inequality to obtain

$$\begin{aligned} \int_0^{1/2} |f'(tb + (1-t)a)|^q dt &= \int_0^{1/2} t^0 |f'(tb + (1-t)a)|^q dt \\ &\leq \left( \int_0^{1/2} t^0 dt \right) \left| f' \left( \frac{\int_0^{1/2} (tb + (1-t)a) dt}{\int_0^{1/2} t^0 dt} \right) \right|^q \\ &= \frac{1}{2} \left| f' \left( 2 \int_0^{1/2} (tb + (1-t)a) dt \right) \right|^q \\ &= \frac{1}{2} \left| f' \left( \frac{b+3a}{4} \right) \right|^q. \end{aligned}$$

Analogously,

$$\int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \leq \frac{1}{2} \left| f' \left( \frac{3b+a}{4} \right) \right|^q.$$

Combining all the obtained inequalities, we get

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)}{6^{\frac{2q-1}{q-1}}} \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \left( \frac{1}{2} \right)^q \left[ \left| f' \left( \frac{3b+a}{4} \right) \right| + \left| f' \left( \frac{b+3a}{4} \right) \right| \right] \\ & \leq (b-a) \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \left[ \left| f' \left( \frac{3b+a}{4} \right) \right| + \left| f' \left( \frac{b+3a}{4} \right) \right| \right], \end{aligned}$$

which completes the proof. ■

**Theorem 10.** Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is  $s$ -concave on  $[a, b]$ , for some fixed  $s \in (0, 1]$  and  $q > 1$ , then the following inequality holds:

$$(2.17) \quad \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq (b-a) 2^{(s-1)/q} \frac{1}{6^{\frac{2q-1}{q-1}}} \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \\ \times \left[ \left| f' \left( \frac{3a+b}{2} \right) \right| + \left| f' \left( \frac{a+3b}{2} \right) \right| \right].$$

*Proof.* We proceed similarly as in the proof of Theorem 9, by using (1.5) instead of Jensen's integral inequality for concave functions. For  $|f'|^q$   $s$ -concave, we have

$$\int_0^{1/2} |f'(tb + (1-t)a)|^q dt \leq 2^{s-1} \left| f' \left( \frac{3a+b}{2} \right) \right|^q,$$

and

$$\int_{1/2}^1 |f'(tb + (1-t)a)|^q dt \leq 2^{s-1} \left| f' \left( \frac{a+3b}{2} \right) \right|^q,$$

so that,

$$\begin{aligned} & \left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) 2^{(s-1)/q} \frac{1}{6^{\frac{2q-1}{q-1}}} \left( \frac{q-1}{2q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \\ & \quad \times \left[ \left| f' \left( \frac{3a+b}{2} \right) \right| + \left| f' \left( \frac{a+3b}{2} \right) \right| \right], \end{aligned}$$

which completes the proof. ■

**Remark 3.**

- (1) In Theorems 7 – 10, if  $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$ , one can obtain new inequalities of midpoint type. However, the details are left to the interested reader.
- (2) All of the above inequalities obviously hold for convex functions. Simply choose  $s = 1$  in each of the results to obtain the desired ones.

## 3. APPLICATIONS TO SPECIAL MEANS

Let  $s \in (0, 1]$  and  $u, v, w \in \mathbb{R}$ . We define a function  $f : [0, \infty) \rightarrow \mathbb{R}$  as

$$f(t) = \begin{cases} u, & t = 0; \\ vt^s + w, & t > 0. \end{cases}$$

If  $v \geq 0$  and  $0 \leq w \leq u$ , then  $f \in K_s^2$  (see [13]). Hence, for  $u = w = 0, v = 1$ , we have  $f : [a, b] \rightarrow \mathbb{R}, f(t) = t^s, f \in K_s^2$ .

In [13], the following result is given:

*Let  $f : I_1 \rightarrow \mathbb{R}_+$  be a non-decreasing and  $s$ -convex function on  $I_1$  and  $g : J \rightarrow I_2 \subseteq I_1$  be a non-negative convex function on  $J$ , then  $f \circ g$  is  $s$ -convex on  $I_1$ .*

A simple consequence of the previous result may be stated as follows:

**Corollary 7.** *Let  $g : I \rightarrow I_1 \subseteq [0, \infty)$  be a non-negative convex function on  $I$ , then  $g^s(x)$  is  $s$ -convex on  $[0, \infty)$ ,  $0 < s < 1$ .*

For arbitrary real numbers  $\alpha, \beta$  ( $\alpha \neq \beta$ ), we consider the following means:

(1) *The arithmetic mean:*

$$A = A(\alpha, \beta) := \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R};$$

(2) *The logarithmic mean:*

$$L = L(\alpha, \beta) := \frac{b - a}{\ln b - \ln a}, \quad \alpha, \beta \in \mathbb{R}, \alpha \neq \beta;$$

(3) *The generalized log-mean:*

$$L_p = L_p(\alpha, \beta) := \left[ \frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}, \alpha, \beta \in \mathbb{R}, \alpha \neq \beta.$$

It is well known that  $L_p$  is monotonic nondecreasing over  $p \in \mathbb{R}$ , with  $L_{-1} := L$  and  $L_0 := I$ . In particular, we have the following inequality  $L \leq A$ .

In the following, some new inequalities are derived for the above means.

(1) Consider  $f : [a, b] \rightarrow \mathbb{R}, (0 < a < b), f(x) = x^s, s \in (0, 1]$ . Then,

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &= L_s^s(a, b), \\ \frac{f(a) + f(b)}{2} &= A(a^s, b^s), \\ f\left(\frac{a+b}{2}\right) &= A^s(a, b). \end{aligned}$$

(a) Using the inequality (2.2), we obtain

$$\begin{aligned} & \left| \frac{1}{3}A(a^s, b^s) + \frac{2}{3}A^s(a, b) - L_s^s(a, b) \right| \\ & \leq s(b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} \left[ |a|^{s-1} + |b|^{s-1} \right]. \end{aligned}$$

For instance, if  $s = 1$  then we get

$$|A(a, b) - L(a, b)| \leq \frac{5}{36} (b - a).$$

(b) Using the inequality (2.4), we have

$$\begin{aligned} & |A^s(a, b) - L_s^s(a, b)| \\ & \leq s(b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} \left[ |a|^{s-1} + |b|^{s-1} \right]. \end{aligned}$$

For instance, if  $s = 1$  then we obtain

$$|A(a, b) - L(a, b)| \leq \frac{5}{72} (b-a).$$

(c) Using the inequality (2.6), we get

$$\begin{aligned} & \left| \frac{1}{3}A(a^s, b^s) + \frac{2}{3}A^s(a, b) - L_s^s(a, b) \right| \\ & \leq (b-a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{s}{(s+1)^{\frac{1}{q}}} \left[ \left( |a^{s-1}|^q + |A^{s-1}(a, b)|^q \right)^{\frac{1}{q}} \right. \\ & \qquad \qquad \qquad \left. + \left( |A^{s-1}(a, b)|^q + |b^{s-1}|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where,  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . For instance, if  $s = 1$  then we have

$$|A(a, b) - L(a, b)| \leq 2(b-a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}}, \quad p > 1.$$

(2) Consider  $f : [a, b] \subseteq (0, \infty) \rightarrow \mathbb{R}$ , ( $0 < a < b$ ),  $f(x) = \frac{1}{x^s} \in K_s^2$  (by Corollary 7),  $s \in (0, 1]$ . Then,

$$\frac{1}{b-a} \int_a^b f(x) dx = L_{-s}^s(a, b),$$

$$\frac{f(a) + f(b)}{2} = A(a^{-s}, b^{-s}),$$

$$f\left(\frac{a+b}{2}\right) = A^{-s}(a, b).$$

(a) Using the inequality (2.2), we obtain

$$\begin{aligned} & \left| \frac{1}{3}A(a^{-s}, b^{-s}) + \frac{2}{3}A^{-s}(a, b) - L_{-s}^s(a, b) \right| \\ & \leq s(b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} \left[ |a|^{-s-1} + |b|^{-s-1} \right]. \end{aligned}$$

For instance, if  $s = 1$  then we get

$$\left| \frac{1}{3}A(a^{-1}, b^{-1}) + \frac{2}{3}A^{-1}(a, b) - L_{-1}(a, b) \right| \leq \frac{5}{36} (b-a) \left[ |a|^{-2} + |b|^{-2} \right].$$

(b) Using the inequality (2.4), we have

$$\begin{aligned} & |A^{-s}(a, b) - L_{-s}^s(a, b)| \\ & \leq s(b-a) \frac{6^{-s} - 9(2)^{-s} + (5)^{s+2}6^{-s} + 3s - 12}{18(s^2 + 3s + 2)} \left[ |a|^{-s-1} + |b|^{-s-1} \right]. \end{aligned}$$

For instance, if  $s = 1$  then we obtain

$$|A^{-1}(a, b) - L_{-1}(a, b)| \leq \frac{5}{72}(b-a) \left[ |a|^{-2} + |b|^{-2} \right].$$

(c) Using the inequality (2.6), we get

$$\begin{aligned} & \left| \frac{1}{3}A(a^{-s}, b^{-s}) + \frac{2}{3}A^{-s}(a, b) - L_{-s}^s(a, b) \right| \\ & \leq (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{s}{(s+1)^{\frac{1}{q}}} \left[ \left( |a^{-s-1}|^q + |A^{-s-1}(a, b)|^q \right)^{\frac{1}{q}} \right. \\ & \qquad \qquad \qquad \left. + \left( |A^{-s-1}(a, b)|^q + |b^{-s-1}|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

where,  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . For instance, if  $s = 1$  then we have

$$\begin{aligned} & \left| \frac{1}{3}A(a^{-1}, b^{-1}) + \frac{2}{3}A^{-1}(a, b) - L_{-1}(a, b) \right| \\ & \leq (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \frac{1}{2^{\frac{1}{q}}} \left[ \left( |a^{-2}|^q + |A^{-2}(a, b)|^q \right)^{\frac{1}{q}} \right. \\ & \qquad \qquad \qquad \left. + \left( |A^{-2}(a, b)|^q + |b^{-2}|^q \right)^{\frac{1}{q}} \right], \quad p > 1. \end{aligned}$$

#### 4. APPLICATIONS TO SOME NUMERICAL QUADRATURE RULES

Using the results of Section 2, we now provide some applications for numerical quadrature rules. Namely, we will consider the Simpson and Midpoint rules.

**4.1. Applications to Simpson's Formula.** Let  $d$  be a division of the interval  $[a, b]$ , i.e.,  $d : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ ,  $h_i = (x_{i+1} - x_i)/2$  and consider the Simpson's formula

$$(4.1) \quad S(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6} (x_{i+1} - x_i).$$

It is well known that if the mapping  $f : [a, b] \rightarrow \mathbb{R}$ , is differentiable such that  $f^{(4)}(x)$  exists on  $(a, b)$  and  $M = \max_{x \in (a, b)} |f^{(4)}(x)| < \infty$ , then

$$(4.2) \quad I = \int_a^b f(x) dx = S(f, d) + E_S(f, d),$$

where the approximation error  $E_S(f, d)$  of the integral  $I$  by Simpson's formula  $S(f, d)$  satisfies

$$(4.3) \quad |E_S(f, d)| \leq \frac{K}{90} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5.$$

It is clear that if the mapping  $f$  is not four times differentiable or the fourth derivative is not bounded on  $(a, b)$ , then (4.2) cannot be applied. In the following we give many different estimations for the remainder term  $E_S(f, d)$  in terms of the first derivative.

**Proposition 1.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then in (4.2), for every division  $d$  of  $[a, b]$ , the following holds:*

$$|E_S(f, d)| \leq \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|].$$

*Proof.* Applying Corollary 1 on the subintervals  $[x_i, x_{i+1}]$ , ( $i = 0, 1, \dots, n-1$ ) of the division  $d$ , we get

$$\begin{aligned} \left| \frac{(x_{i+1} - x_i)}{3} \left( f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1}) \right) - \int_{x_i}^{x_{i+1}} f(x) dx \right| \\ \leq \frac{5(x_{i+1} - x_i)^2}{72} [|f'(x_i)| + |f'(x_{i+1})|]. \end{aligned}$$

Summing over  $i$  from 0 to  $n-1$  and taking into account that  $|f'|$  is convex, we deduce, by the triangle inequality, that

$$\left| S(f, d) - \int_a^b f(x) dx \right| \leq \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|].$$

which completes the proof. ■

**Proposition 2.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^{p/(p-1)}$  is convex on  $[a, b]$ ,  $p > 1$ , then in (4.2), for every division  $d$  of  $[a, b]$ , the following holds:*

$$\begin{aligned} |E_S(f, d)| \leq 2^{-\frac{1}{q}} \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[ \left( |f'(x_i)|^q + \left| f' \left( \frac{x_i + x_{i+1}}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left( \left| f' \left( \frac{x_i + x_{i+1}}{2} \right) \right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

*Proof.* The proof is similar to that of Proposition 1, using the proof of Corollary 4. ■

**Proposition 3.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some fixed  $q \geq 1$ , then in (4.2), for every division  $d$  of  $[a, b]$ , the following holds:*

$$|E_S(f, d)| \leq \frac{5}{72} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[ \left| f' \left( \frac{29x_{i+1} + 61x_i}{90} \right) \right| + \left| f' \left( \frac{61x_{i+1} + 29x_i}{90} \right) \right| \right].$$

*Proof.* The proof is similar to that of Proposition 1, using the proof of Theorem 8. ■

**Proposition 4.** *Let  $f : I \subset [0, \infty) \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  such that  $f' \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $|f'|^q$  is concave on  $[a, b]$ , for some*

fixed  $q > 1$ , then in (4.2), for every division  $d$  of  $[a, b]$ , the following holds:

$$|E_S(f, d)| \leq \left( \frac{2q-1}{q-1} \right) \left( 2^{\frac{2q-1}{q-1}} + 1 \right) \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[ \left| f' \left( \frac{3x_{i+1} + x_i}{4} \right) \right| + \left| f' \left( \frac{x_{i+1} + 3x_i}{4} \right) \right| \right].$$

*Proof.* The proof is similar to that of Proposition 1, using the proof of Theorem 9. ■

**4.2. Applications to the Midpoint Formula.** Let  $d$  be a division of the interval  $[a, b]$ , i.e.,  $d : a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ , and consider the midpoint formula

$$(4.4) \quad M(f, d) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_i + x_{i+1}}{2} \right).$$

It is well known that if the mapping  $f : [a, b] \rightarrow \mathbb{R}$ , is differentiable such that  $f''(x)$  exists on  $(a, b)$  and  $K = \sup_{x \in (a, b)} |f''(x)| < \infty$ , then

$$(4.5) \quad I = \int_a^b f(x) dx = M(f, d) + E_M(f, d),$$

where the approximation error  $E_M(f, d)$  of the integral  $I$  by the midpoint formula  $M(f, d)$  satisfies

$$(4.6) \quad |E_M(f, d)| \leq \frac{\tilde{K}}{24} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.$$

In the following, we propose some new estimates for the remainder term  $E_M(f, d)$  in terms of the first derivative which are better than the estimations of [18].

**Proposition 5.** *Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$ , then in (4.5), for every division  $d$  of  $[a, b]$ , the following holds:*

$$|E_M(f, d)| \leq \frac{5}{72} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|]$$

*Proof.* Applying Corollary 3 on the subintervals  $[x_i, x_{i+1}]$ , ( $i = 0, 1, \dots, n-1$ ) of the division  $d$ , we get

$$\left| (x_{i+1} - x_i) f \left( \frac{x_i + x_{i+1}}{2} \right) - \int_{x_i}^{x_{i+1}} f(x) dx \right| \\ \leq \frac{5(x_{i+1} - x_i)^2}{72} [|f'(x_i)| + |f'(x_{i+1})|].$$

Summing over  $i$  from 0 to  $n-1$  and taking into account that  $|f'|$  is convex, we deduce that

$$|E_M(f, d)| \leq \frac{5}{72} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)| + |f'(x_{i+1})|]$$

which completes the proof. ■



**Proposition 6.** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If  $|f'|^{p/(p-1)}$  is convex on  $[a, b]$ ,  $p > 1$ , then in (4.5), for every division  $d$  of  $[a, b]$ , the following holds:

$$|E_M(f, d)| \leq 2^{-\frac{1}{q}} \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \\ \times \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[ \left( |f'(x_i)|^q + \left| f' \left( \frac{x_i + x_{i+1}}{2} \right) \right|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left( \left| f' \left( \frac{x_i + x_{i+1}}{2} \right) \right|^q + |f'(x_{i+1})|^q \right)^{\frac{1}{q}} \right].$$

*Proof.* The proof is similar to that of Proposition 5, using Corollary 6. ■

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