SUBADDITIVITY OF SOME FUNCTIONALS ASSOCIATED TO JENSEN'S INEQUALITY WITH APPLICATIONS

S.S. DRAGOMIR, Y.J. CHO*, AND J.K. KIM

Abstract. Some new results related to Jensen's celebrated inequality for convex functions defined on convex sets in linear spaces are given. Applications for the arithmetic mean-geometric mean inequality are provided as well.

1. Introduction

Let C be a convex subset of the linear space X and f a convex function on C. If I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C, p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$, then we have

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{1.1}$$

which is well known in the literature as *Jensen's inequality*.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the generalised triangle inequality, the arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

In order to simplify the presentation, we introduce the following notations (see also [14]):

$$\begin{aligned} F(C,\mathbb{R}) &:= \text{the linear space of all real functions on } C, \\ F^{+}(C,\mathbb{R}) &:= \{f \in F(C,\mathbb{R}) : f(x) > 0 \text{ for all } x \in C\}, \\ P_f(\mathbb{N}) &:= \{I \subset \mathbb{N}\colon I \text{ is finite}\}, \\ J(\mathbb{R}) &:= \left\{p = \{p_i\}_{i\in\mathbb{N}},\ p_i \in \mathbb{R} \text{ are such that } P_I \neq 0 \text{ for all } I \in P_f(\mathbb{N})\right\}, \end{aligned}$$

and

$$\begin{aligned} J^{+}(\mathbb{R}) &:= \{p \in J(\mathbb{R}) : p_i \geq 0 \text{ for all } i \in \mathbb{N}\}, \\ J_{*}(C) &:= \left\{x = \{x_i\}_{i\in\mathbb{N}} : x_i \in C \text{ for all } i \in \mathbb{N}\right\} \end{aligned}$$

and

$$Conv(C,\mathbb{R}) := \text{ the cone of all convex functions defined on } C,$$

respectively.

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*Corresponding author.

In [14] the authors considered the following functional associated with the Jensen inequality:

$$J(f,I,p,x) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right), \tag{1.2}$$

where $f \in F(C,\mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$. They established some quasi-linearity and monotonicity properties and applied the obtained results for norm and means inequalities.

The following result concerning the properties of the functional $J(f,I,\cdot,x)$ as a *function of weights* holds (see [14, Theorem 2.4]):

Theorem 1. *Let* $f \in Conv(C,\mathbb{R})$, $I \in P_f(\mathbb{N})$ *and* $x \in J_*(C)$.

(i) *If* $p,q \in J^+(\mathbb{R})$ *then*

$$J(f,I,p+q,x) \geq J(f,I,p,x) + J(f,I,q,x)\ (\geq 0) \tag{1.3}$$

i.e., $J(f,I,\cdot,x)$ *is superadditive on* $J^+(\mathbb{R})$*;*

(ii) *If* $p,q \in J^+(\mathbb{R})$ *with* $p \geq q$*, meaning that* $p_i \geq q_i$ *for each* $i \in \mathbb{N}$*, then*

$$J(f,I,p,x) \geq J(f,I,q,x)\ (\geq 0) \tag{1.4}$$

i.e., $J(f,I,\cdot,x)$ *is monotonic nondecreasing on* $J^+(\mathbb{R})$.

The behavior of this functional as an *index set function* is incorporated in the following (see [14, Theorem 2.1]):

Theorem 2. *Let* $f \in Conv(C,\mathbb{R})$, $p \in J^+(\mathbb{R})$ *and* $x \in J_*(C)$.

(i) *If* $I,H \in P_f(\mathbb{N})$ *with* $I \cap H = \varnothing$*, then*

$$J(f,I\cup H,p,x) \geq J(f,I,p,x) + J(f,H,p,x)\ (\geq 0), \tag{1.5}$$

i.e., $J(f,\cdot,p,x)$ *is superadditive as an index set function on* $P_f(\mathbb{N})$*;*

(ii) *If* $I,H \in P_f(\mathbb{N})$ *with* $H \subset I$*, then*

$$J(f,I,p,x) \geq J(f,H,p,x)\ (\geq 0), \tag{1.6}$$

i.e., $J(f,\cdot,p,x)$ *is monotonic nondecreasing as an index set function on* $P_f(\mathbb{N})$.

As pointed out in [14], the above Theorem 2 is a generalisation of the Vasić-Mijalković result for convex functions of a real variable obtained in [16] and therefore creates the possibility to obtain vectorial inequalities as well.

For applications of the above results to logarithmic convex functions, to norm inequalities, in relation with the arithmetic mean-geometric mean inequality and with other classical results, see [14].

Motivated by the above results, we introduce in the present paper another functional associated to Jensen's discrete inequality, establish its subadditivity properties as both a function of weights and an index set function and use it for some particular cases that provide inequalities of interest. Applications related to the arithmetic mean - geometric mean celebrated inequality are provided as well.

2. Some Subadditivity Properties for the Weights

We consider the more general functional

$$D(f,I,p,x;\Psi):=P_I\Psi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right], \tag{2.1}$$

where $f\in Conv(C,\mathbb{R})$, $I\in P_f(\mathbb{N})$, $p\in J^+(\mathbb{R})$, $x\in J_*(C)$ and $\Psi:[0,\infty)\to\mathbb{R}$ is a function whose properties will determine the behavior of the functional D as follows. Obviously, for $\Psi(t)=t$ we recapture from D the functional J considered in [14].

First of all we observe that, by Jensen's inequality, the functional D is well defined and *positive homogeneous* in the third variable, i.e.,

$$D(f,I,\alpha p,x;\Psi)=\alpha D(f,I,p,x;\Psi),$$

for any $\alpha>0$ and $p\in J^+(\mathbb{R})$.

The following result concerning the subadditivity of the functional D as a function of weights holds:

Theorem 3. *Let $f\in Conv(C,\mathbb{R}), I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Psi:[0,\infty)\to\mathbb{R}$ is monotonic nonincreasing and convex where it is defined. If $p,q\in J^+(\mathbb{R})$ then*

$$D(f,I,p+q,x;\Psi)\le D(f,I,p,x;\Psi)+D(f,I,q,x;\Psi), \tag{2.2}$$

i.e., D is subadditive as a function of weights.

Proof. Let $p,q\in J^+(\mathbb{R})$. It is easy to see that, by the convexity of the function f on C, we have

$$\begin{aligned}
&\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right) \\
&=\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)\right)}{P_I+Q_I} \\
&\quad -f\left(\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)}{P_I+Q_I}\right) \\
&\ge\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)\right)}{P_I+Q_I} \\
&\quad -\frac{P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_If\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)}{P_I+Q_I} \\
&=\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I} \\
&\quad +\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}.
\end{aligned} \tag{2.3}$$

Since Ψ is monotonic nonincreasing, then by (2.3) we have

$$\begin{aligned}(2.4)\quad &\Psi\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right]\\ &\leq\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\right.\\ &\left.+\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}\right\}.\end{aligned}$$

Now, on utilising the convexity property of Ψ we also have

$$\begin{aligned}(2.5)\quad &\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\right.\\ &\left.+\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}\right\}\\ &\leq\frac{P_I\Psi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\\ &+\frac{Q_I\Psi\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}.\end{aligned}$$

Finally, on making use of (2.4) and (2.5), we deduce the desired inequality (2.2). ∎

Obviously, there are many examples of functions $\Psi:[0,\infty)\to\mathbb{R}$ that are monotonically decreasing and convex on the interval $[0,\infty)$. In what follows we give some examples that are of interest.

Example 1. *Consider the function $\Psi:[0,\infty)\to(0,\infty)$ defined by $\Psi(t)=\exp(-t)$. Obviously this function is strictly decreasing and strictly convex on the interval $[0,\infty)$ and we can consider the functional*

$$(2.6)\quad E(f,I,p,x):=D(f,I,p,x;\exp(-\cdot))=\frac{P_I\exp\left[f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[p_if(x_i)\right]\right\}^{\frac{1}{P_I}}}.$$

Since the functional $E(f,I,\cdot,x)$ is subadditive, then we can state the following interesting inequality for convex functions

$$\begin{aligned}(2.7)\quad &\frac{(P_I+Q_I)\exp\left[f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[(p_i+q_i)f(x_i)\right]\right\}^{\frac{1}{P_I+Q_I}}}\\ &\leq\frac{P_I\exp\left[f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[p_if(x_i)\right]\right\}^{\frac{1}{P_I}}}+\frac{Q_I\exp\left[f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[q_if(x_i)\right]\right\}^{\frac{1}{Q_I}}}\end{aligned}$$

for any $p,q\in J^{+}(\mathbb{R})$.

Example 2. *Now assume that* $f \in Conv(C, \mathbb{R})$ *and* $x \in J_*(C)$ *are selected such that*

$$\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) > f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)$$

for any $I \in P_f(\mathbb{N})$ *with* $card(I) \geq 2$ *and* $p \in J^+(\mathbb{R})$ *(notice that is enough to assume that* f *is strictly convex and* x *is not constant). If we consider the function* $\Psi : (0,\infty) \to (0,\infty)$ *defined by* $\Psi(t) = t^{-\alpha}$ *with* $\alpha > 0$, *then obviously this function is strictly decreasing and strictly convex on the interval* $(0,\infty)$ *and we can consider the functional*

$$W(f, I, p, x) := D\left(f, I, p, x; (\cdot)^{-\alpha}\right) = \frac{P_I}{\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{\alpha}}. \tag{2.8}$$

Since the functional $E(f, I, \cdot, x)$ *is subadditive, we can state the following interesting inequality for convex functions*

$$\frac{P_I + Q_I}{\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i) f(x_i) - f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i) x_i\right)\right]^{\alpha}} \leq \frac{P_I}{\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^{\alpha}} + \frac{Q_I}{\left[\frac{1}{Q_I}\sum_{i\in I} q_i f(x_i) - f\left(\frac{1}{Q_I}\sum_{i\in I} q_i x_i\right)\right]^{\alpha}} \tag{2.9}$$

for any $p, q \in J^+(\mathbb{R})$ *such that the involved denominators are not zero.*

Corollary 1. *Let* $f \in Conv(C, \mathbb{R}), I \in P_f(\mathbb{N})$ *and* $x \in J_*(C)$. *Assume that* $\Xi : [0,\infty) \to (0,\infty)$. *We define the new functional*

$$M(f, I, p, x; \Xi) := \left\{\Xi\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]\right\}^{P_I}. \tag{2.10}$$

If $\Xi : [0,\infty) \to (0,\infty)$ *is monotonic nonincreasing and logarithmic convex, i.e.* $\ln(\Xi)$ *is a convex function, then for any* $p, q \in J^+(\mathbb{R})$ *we have*

$$M(f, I, p+q, x; \Xi) \leq M(f, I, p, x; \Xi) \cdot M(f, I, q, x; \Xi), \tag{2.11}$$

i.e., the functional is submultiplicative as a function of weights.

Proof. Consider the function $\Psi = \ln(\Xi)$ which is convex and, obviously

$$D(f, I, p, x; \Psi) = \ln M(f, I, p, x; \Xi).$$

The inequality (2.11) follows now by (2.2) and the details are omitted. ∎

Example 3. *We consider the Dirichlet series generated by a nonnegative sequence* $a_n, n \geq 1$ *namely* $\delta : (0,\infty) \to (0,\infty)$ *given by*

$$\delta(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1}}. \tag{2.12}$$

An important example of such series is the Zeta function defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \textit{for } s > 1.$$

It is known that the function δ *is monotonic nondecreasing and logarithmic convex on* $(0,\infty)$ *(see for instance* [3] *). Therefore, for any Dirichlet series of the form (2.12) we have the inequalities*

$$\left\{\delta\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right]\right\}^{P_I+Q_I} \le \left\{\delta\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]\right\}^{P_I} \times \left\{\delta\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]\right\}^{Q_I} \tag{2.13}$$

for any $p,q\in J^{+}(\mathbb{R})$.

3. Some Subadditivity Properties for the Index

The following result concerning the superadditivity and monotonicity of the functional D as an index set function holds:

Theorem 4. *Let* $f\in Conv(C,\mathbb{R})$, $p\in J^{+}(\mathbb{R})$ *and* $x\in J_{*}(C)$. *Assume that* $\Psi:[0,\infty)\to\mathbb{R}$ *is monotonic nonincreasing and convex where it is defined. If* $I,H\in P_f(\mathbb{N})$ *with* $I\cap H=\varnothing$, *then*

$$D(f,I\cup H,p,x;\Psi)\le D(f,I,p,x;\Psi)+D(f,H,p,x;\Psi), \tag{3.1}$$

i.e., $D(f,\cdot,p,x;\Psi)$ *is subadditive as an index set function on* $P_f(\mathbb{N})$.

Proof. Let $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$. By the convexity of the function f on C, we have successively

$$\begin{aligned}
&\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_kf(x_k)-f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_kx_k\right) \\
&=\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)\right)}{P_I+P_H} \\
&\quad -f\left(\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)}{P_I+P_H}\right) \\
&\ge\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)\right)}{P_I+P_H} \\
&\quad -\frac{P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+P_Hf\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)}{P_I+P_H}
\end{aligned} \tag{3.2}$$

$$= \frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+P_H} + \frac{P_H\left[\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)\right]}{P_I+P_H}.$$

Since Ψ is monotonic nonincreasing, then by (3.2) we have

$$\Psi\left[\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_kf(x_k)-f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_kx_k\right)\right] \leq \Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+P_H} + \frac{P_H\left[\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)\right]}{P_I+P_H}\right\}. \tag{3.3}$$

Utilising the convexity of the function Ψ we also have that

$$\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+P_H} + \frac{P_H\left[\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)\right]}{P_I+P_H}\right\} \leq \frac{P_I\Psi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+P_H} + \frac{P_H\Psi\left[\frac{1}{P_H}\sum_{j\in H}p_jf(x_j)-f\left(\frac{1}{P_H}\sum_{j\in H}p_jx_j\right)\right]}{P_I+P_H}, \tag{3.4}$$

which together with (3.3) produces the desired result (3.1) ∎

Example 4. *With the assumptions in Example 1 and utilising (3.1), we have the inequality*

$$\frac{P_{I\cup H}\exp\left[f\left(\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_ix_i\right)\right]}{\left\{\prod_{i\in I\cup H}\exp\left[p_if(x_i)\right]\right\}^{\frac{1}{P_{I\cup H}}}} \leq \frac{P_I\exp\left[f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[p_if(x_i)\right]\right\}^{\frac{1}{P_I}}} + \frac{P_H\exp\left[f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]}{\left\{\prod_{i\in H}\exp\left[p_if(x_i)\right]\right\}^{\frac{1}{P_H}}}, \tag{3.5}$$

for any $I,H\in P_f(\mathbb{N})$ *with* $I\cap H=\varnothing$.

Example 5. *With the assumptions in Example 1 and making use of (3.1), we also have the inequality*

$$\frac{P_{I\cup H}}{\left[\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_if(x_i)-f\left(\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_ix_i\right)\right]^\alpha} \le \frac{P_I}{\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^\alpha} + \frac{P_H}{\left[\frac{1}{P_H}\sum_{i\in H}p_if(x_i)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]^\alpha} \tag{3.6}$$

for any $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$ and such that the involved denominators are not zero.

If we use the superadditivity property, then we can state the following result as well:

Corollary 2. *Let $f\in Conv(C,\mathbb{R}),p\in J^+(\mathbb{R})$ and $x\in J_*(C)$. Assume that $\Psi:[0,\infty)\to\mathbb{R}$ is monotonic nonincreasing and convex where it is defined. Then*

$$P_{2n}\Psi\left[\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_if(x_i)-f\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_ix_i\right)\right] \ge \sum_{i=1}^n p_{2i}\Psi\left[\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i}f(x_{2i})-f\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i}x_{2i}\right)\right] + \sum_{i=1}^n p_{2i-1}\Psi\left[\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^n p_{2i-1}f(x_{2i-1})-f\left(\frac{1}{\sum_{i=1}^n p_{2i-1}}\sum_{i=1}^n p_{2i-1}x_{2i-1}\right)\right] \tag{3.7}$$

and

$$P_{2n+1}\Psi\left[\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_if(x_i)-f\left(\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_ix_i\right)\right] \ge \sum_{i=1}^n p_{2i}\Psi\left[\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i}f(x_{2i})-f\left(\frac{1}{\sum_{i=1}^n p_{2i}}\sum_{i=1}^n p_{2i}x_{2i}\right)\right] + \sum_{i=1}^n p_{2i+1}\Psi\left[\frac{1}{\sum_{i=1}^n p_{2i+1}}\sum_{i=1}^n p_{2i+1}f(x_{2i+1})-f\left(\frac{1}{\sum_{i=1}^n p_{2i+1}}\sum_{i=1}^n p_{2i+1}x_{2i+1}\right)\right], \tag{3.8}$$

where $P_{2n}:=\sum_{i=1}^{2n}p_i$ and $P_{2n+1}:=\sum_{i=1}^{2n+1}p_i$.

The following submultiplicity result also holds:

Corollary 3. *Let $f\in Conv(C,\mathbb{R}),I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Xi:[0,\infty)\to(0,\infty)$ is monotonic nonincreasing and logarithmic convex. If $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$, then*

$$M(f,I\cup H,p,x;\Xi)\le M(f,I,p,x;\Xi)\cdot M(f,H,p,x;\Xi), \tag{3.9}$$

i.e., $M(f,\cdot,p,x;\Xi)$ is submultiplicative as an index set function on $P_f(\mathbb{N})$;

4. Applications for the Arithmetic Mean-Geometric Mean Inequality

For two sequences of positive numbers p and x, we use the notations

$$A(p,x,I) := \frac{1}{P_I}\sum_{i\in I} p_i x_i \quad \text{and} \quad G(p,x,I) := \left(\prod_{i\in I} x_i^{p_i}\right)^{\frac{1}{P_I}},$$

where I is a finite set of indices and $A(p,x,I)$ is the *arithmetic mean* while $G(p,x,I)$ is the *geometric mean* of the numbers x_i with the weights p_i, $i \in I$.

It is well known that

$$A(p,x,I) \geq G(p,x,I), \tag{4.1}$$

which is known in the literature as the *arithmetic mean-geometric mean inequality.* For various results related to this inequality we recommend the monograph [2] and the references therein.

For the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t) := -\ln(t)$, consider the functional

$$L(I,p,x;\Psi) := D(-\ln(\cdot), I, p, x; \Psi) := P_I \Psi\left[\ln\left(\frac{A(p,x,I)}{G(p,x,I)}\right)\right]. \tag{4.2}$$

We can state the following.

Proposition 1. *Let $f \in Conv(C,\mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Psi:[0,\infty)\to\mathbb{R}$ is monotonic nonincreasing and convex where it is defined.*

(i) *If $p, q \in J^+(\mathbb{R})$, then*

$$L(I,p+q,x;\Psi) \leq L(I,p,x;\Psi) + L(I,q,x;\Psi) \tag{4.3}$$

i.e., L is subadditive as a function of weights.

(ii) *If $I, H \in P_f(\mathbb{N})$ with $I\cap H = \varnothing$, then*

$$L(I\cup H,p,x;\Psi) \leq L(I,p,x;\Psi) + L(H,p,x;\Psi), \tag{4.4}$$

i.e., L is subadditive as an index set function on $P_f(\mathbb{N})$.

Utilising these inequalities, we can state the following results concerning the arithmetic and geometric means:

Example 6. *Consider the function $\Psi:[0,\infty)\to(0,\infty)$ defined by $\Psi(t) = \exp(-t)$. Obviously this function is strictly decreasing and strictly convex on the interval $[0,\infty)$ and we can consider the functional*

$$L_e(I,p,x) := L(I,p,x;\exp(-\cdot)) = \frac{P_I G(p,x,I)}{A(p,x,I)}. \tag{4.5}$$

By Proposition 1 above, we have that L_e is both additive as a weights and index set functional.

We can give the following example as well:

Example 7. *If we consider the function $\Psi:(0,\infty)\to(0,\infty)$ defined by $\Psi(t) = t^{-\alpha}$ with $\alpha > 0$, then obviously this function is strictly decreasing and strictly convex on the interval $(0,\infty)$ and we can consider the functional*

$$W_{\ln,\alpha}(I,p,x) := L\left(I,p,x;(\cdot)^{-\alpha}\right) = P_I\left[\ln\left(\frac{A(p,x,I)}{G(p,x,I)}\right)\right]^{-\alpha}. \tag{4.6}$$

By the above Proposition 1 we have that $W_{\ln,\alpha}$ is both additive as a weights and index set functional.

Now, for positive sequences x we introduce the notation

$$G(p, x^x, I) := \left(\prod_{i\in I} x_i^{p_i x_i}\right)^{\frac{1}{P_I}}, \tag{4.7}$$

which is the geometric mean of the sequence having the terms $x_i^{x_i}$, $i \in I$.

For the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) := t \ln(t)$, consider the functional

$$S(I, p, x; \Psi) := D(\cdot \ln(\cdot), I, p, x; \Psi) := P_I \Psi \left[\ln\left(\frac{G(p, x^x, I)}{A(p, x, I)^{A(p,x,I)}}\right)\right]. \tag{4.8}$$

We can state the following.

Proposition 2. *Let $f \in Conv(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Psi : [0, \infty) \to \mathbb{R}$ is monotonic nonincreasing and convex where it is defined.*

(i) *If $p, q \in J^+(\mathbb{R})$, then*

$$S(I, p+q, x; \Psi) \le S(I, p, x; \Psi) + S(I, q, x; \Psi), \tag{4.9}$$

i.e., S is subadditive as a function of weights.

(ii) *If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \varnothing$, then*

$$S(I \cup H, p, x; \Psi) \le S(I, p, x; \Psi) + S(H, p, x; \Psi), \tag{4.10}$$

i.e., S is subadditive as an index set function on $P_f(\mathbb{N})$.

Remark 1. *For the function $\Psi : [0, \infty) \to (0, \infty)$ defined by $\Psi(t) = \exp(-t)$ we can consider the functional*

$$S_e(I, p, x) := S(I, p, x; \exp(-\cdot)) = \frac{P_I A(p, x, I)^{A(p,x,I)}}{G(p, x^x, I)}. \tag{4.11}$$

By the above Proposition 2 we have that S_e is both additive as a weights and index set functional.

For the function $\Psi : (0, \infty) \to (0, \infty)$ defined by $\Psi(t) = t^{-\alpha}$ with $\alpha > 0$, we can also consider the functional

$$Z_{\ln,\alpha}(I, p, x) := S\left(I, p, x; (\cdot)^{-\alpha}\right) = P_I \left[\ln\left(\frac{G(p, x^x, I)}{A(p, x, I)^{A(p,x,I)}}\right)\right]^{-\alpha}. \tag{4.12}$$

By the above Proposition 2 we have that $Z_{\ln,\alpha}$ is both additive as a weights and index set functional.

The interested reader can consider other examples of functions f and Ψ and derive functionals that are associated with the Ky Fan, triangle or other inequalities that can be obtained from the Jensen result. However, the details are not presented here.

References

[1] R.P. Agarwal and S.S. Dragomir, The property of supermultiplicity for some classical inequalities and applications, *Comput. Math. Appl.,* **35** (1998), no. 6, 105-118.

[2] P. Bullen, *Handbook of Means and their Inequalities*, Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and their inequalities, Reidel, Dordrecht; MR0947142]. Mathematics and its Applications, 560. Kluwer Academic Publishers Group, Dordrecht, 2003. xxviii+537 pp.

[3] P. Cerone and S.S. Dragomir, Some convexity properties of Dirichlet series with positive terms, *Math. Nachr.,* **282**(2009), No. 7, 964-975

[4] S.S. Dragomir, An improvement of Jensen's inequality, *Bull. Math. Soc. Sci. Math. Roumanie*, **34**(**82**) (1990), No. 4, 291-296.
[5] S.S. Dragomir, Some refinements of Ky Fan's inequality, *J. Math. Anal. Appl.*, **163**(2) (1992), 317-321.
[6] S.S. Dragomir, Some refinements of Jensen's inequality, *J. Math. Anal. Appl.*, **168**(2) (1992), 518-522.
[7] S.S. Dragomir, A further improvement of Jensen's inequality, *Tamkang J. Math.*, **25**(1) (1994), 29-36.
[8] S.S. Dragomir, A new improvement of Jensen's inequality, *Indian J. Pure and Appl. Math.*, **26**(10) (1995), 959-968.
[9] S.S. Dragomir, Bounds for the normalised Jensen functional. *Bull. Austral. Math. Soc.* **74** (2006), no. 3, 471-478.
[10] S.S. Dragomir, A refinement of Jensen's inequality with applications for f-divergence measures, Preprint, *Res. Rep. Coll.* **10**(2007), Supp., Article 15.
[11] S.S. Dragomir, Refinements of the Jensen integral inequality, *in preparation.*
[12] S.S. Dragomir and N.M. Ionescu, Some converse of Jensen's inequality and applications, *Rev. Anal. Numér. Théor. Approx.,* **23** (1994), no. 1, 71-78.
[13] S.S. Dragomir and C.J. Goh, A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Mathl. Comput. Modelling*, **24**(1996), No. 2, 1-11.
[14] S.S. Dragomir, J. Pečarić and L.E. Persson, Properties of some functionals related to Jensen's inequality, *Acta Math. Hung.*, **70**(1-2) (1996), 129-143.
[15] J. Pečarić and S.S. Dragomir, A refinements of Jensen inequality and applications, *Studia Univ. Babeş-Bolyai, Mathematica,* **24**(1) (1989), 15-19.
[16] P. M. Vasić and Ž. Mijajlović, On an index set function connected with Jensen inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.,* No. **544**–**576** (1976), 110-112.

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/rgmia/dragomir/

Department of Mathematical Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: yjcho@nongae.gsnu.ac.kr

Department of Mathematics Education, Kyungnam University, Masan 631-701, Korea
E-mail address: jungkyuk@kyungnam.ac.kr