SCHUR-GEOMETRIC CONCAVITY FOR DIFFERENCE OF SOME MEANS

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Abstract. The Schur-geometric concavity in $(0,\infty)\times(0,\infty)$ for the difference of some famous means such as arithmetic mean, geometric mean, harmonic mean, root-square mean, etc. is discussed. And some inequalities related to the difference of means are obtained.

1. Introduction

Recently, the following chain of inequalities for the binary means is given in [1]:

$$H(a,b) \le G(a,b) \le H_1(a,b) \le H_3(a,b) \le H_2(a,b) \le A(a,b) \le S(a,b), \qquad (1)$$

where

$$H(a,b) = \frac{2ab}{a+b},$$

$$G(a,b) = \sqrt{ab},$$

$$N_1(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \frac{A(a,b)+G(a,b)}{2},$$

$$N_3(a,b) = \frac{a+\sqrt{ab}+b}{3},$$

$$N_2(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)$$

$$A(a,b) = \frac{a+b}{2},$$

$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}},$$

The means, $H(a,b), G(a,b), A(a,b)$ and $S(a,b)$ are harmonic, geometric, arithmetic, root-square, square-root and Heron means respectively.

Throughout the paper we assume that the set of the real number, the nonnegative real number, the nonpositive real number and the positive real number by $\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$ and $\mathbb{R}_{++}$ respectively.

Let $(s,t) \in \mathbb{R}^2, (x,y) \in \mathbb{R}^2_{++}$. The Gini mean of (x,y) is defined in [1] and [2, p. 44] as

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2000 *Mathematics Subject Classification.* Primary 26B25, 26E60, 26D20.

Key words and phrases. arithmetic mean, geometric mean, harmonic mean, root-square mean, square-root mean, Heron mean, difference, Schur-geometric concavity, inequality.

The first author was supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM200611417009).

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

$$G(r,s;x,y)=\begin{cases}\left(\dfrac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)}, & r\neq s,\\ \exp\left(\dfrac{x^s\ln x+y^s\ln y}{x^r+y^r}\right), & r=s.\end{cases}$$

Clearly, $G(0,-1;x,y)$ is the harmonic mean, $G(0,0;x,y)$ is the geometric mean, $G(1,0;x,y)$ is the arithmetic mean.

The extended mean (or Stolarsky mean) of (x,y) is defined in [2, p. 43] as

$$E(r,s;x,y)=\begin{cases}\left(\dfrac{r}{s}\cdot\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0,\\ \left(\dfrac{1}{r}\cdot\dfrac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, & r(x-y)\neq 0;\\ \dfrac{1}{e^{1/r}}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, & r(x-y)\neq 0;\\ \sqrt{xy}, & x\neq y;\\ x, & x=y.\end{cases}$$

The Schur-convexities of the extended mean $E(r,s;x,y)$ with (r,s) and (x,y) were presented in [4, 5] as follows.

Theorem B ([4])**.** *For fixed $\in\mathbb{R}^2_{++}$ with $x\neq y$, $E(r,s;x,y)$ is Schur-concave on $\mathbb{R}^2_{+}$ and Schur-convex on $\mathbb{R}^2_{-}$ with (r,s).*

Theorem C ([5])**.** *For fixed $(r,s)\in\mathbb{R}^2$,*

- *(i) if $2<2r<s$ or $2\le 2s\le r$, then $E(r,s;x,y)$ is Schur-convex on $\mathbb{R}^2_{++}$ with (x,y),*
- *(ii) if $(r,s)\in\{r<s\le 2r, 0<r\le 1\}\cup\{s<r\le 2s, 0<s\le 1\}\cup\{0<s<r\le 1\}\cup\{0<r<s\le 1\}\cup\{s\le 2r<0\}\cup\{r\le 2s<0\}$, then $E(r,s;x,y)$ is Schur-concave on $\mathbb{R}^2_{++}$ with (x,y).*

For more information on the extended mean values and Gini mean, please refer to [3, 4, 5, 6, 7, 8, 9, 10, **?**] and the references therein.

In this paper, the monotonicity and the Schur-convexity with parameters $(s,t)\in\mathbb{R}^2$ for fixed (x,y) and the Schur-convexity and the Schur-geometrically convexity with variables $(x,y)\in\mathbb{R}^2_{++}$ for fixed (s,t) of Gini mean $G(r,s;x,y)$ are discussed. Moreover, some new inequalities are obtained.

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 1 ([11, 12])**.** Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

- *(i)* $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\ge\cdots\ge x_{[n]}$ and $y_{[1]}\ge\cdots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
- *(ii)* $\boldsymbol{x}\ge\boldsymbol{y}$ means $x_i\ge y_i$ for all $i=1,2,\ldots,n$. let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\ge\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\ge\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.
- *(iii)* $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(*iv*) let $\Omega\subset\mathbb{R}^n$, φ: $\Omega\to\mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x}\prec\boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x})\le\varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2 ([13, 14])**.** Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{R}^n_{++}$.

(*i*) $\Omega\subset\mathbb{R}^n_{++}$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta,\dots,x_n^\alpha y_n^\beta)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(*ii*) let $\Omega\subset\mathbb{R}^n_{++}$, φ: $\Omega\to\mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1,\dots,\ln x_n)\prec(\ln y_1,\dots,\ln y_n)$ on Ω implies $\varphi(\boldsymbol{x})\le\varphi(\boldsymbol{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Lemma 1 ([11, p. 38])**.** *A function $\varphi(\boldsymbol{x})$ is increasing if and only if $\nabla\varphi(\boldsymbol{x})\ge 0$ for $\boldsymbol{x}\in\Omega$, where $\Omega\subset\mathbb{R}^n$ is an open set, $\varphi:\Omega\to R$ is differentiable, and*

$$\nabla\varphi(\boldsymbol{x})=\left(\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1},\dots,\frac{\partial\varphi(\boldsymbol{x})}{\partial x_n}\right)\in\mathbb{R}^n.$$

Lemma 2 ([11, p. 58])**.** *Let $\Omega\subset\mathbb{R}^n$ is symmetric with respect to permutations and convex set, and has a nonempty interior set Ω^0 . Let $\varphi:\Omega\to\mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$, if and only if it is symmetric on Ω and if*

$$(x_1-x_2)\left(\frac{\partial\varphi}{\partial x_1}-\frac{\partial\varphi}{\partial x_2}\right)\ge 0(\le 0)$$

holds for any $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)\in\Omega^0$.

Lemma 3 ([13, p. 108])**.** *Let $\Omega\subset\mathbb{R}^n_{++}$ is a symmetric with respect to permutations and geometrically convex set, and has a nonempty interior set Ω^0, Let $\varphi:\Omega\to\mathbb{R}_+$ is continuous on Ω and differentiable in Ω^0.Then φ is the Schur-geometrically convex (Schur-geometrically concave) function if φ is symmetric on Ω and*

$$(\ln x_1-\ln x_2)\left(x_1\frac{\partial\varphi}{\partial x_1}-x_2\frac{\partial\varphi}{\partial x_2}\right)\ge 0(\le 0)$$

holds for any $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)\in\Omega^0$.

Lemma 4 ([**?**, 5])**.** *Let I be an interval with nonempty interior on $\mathbb{R}$ and f be a continuous function on I. The arithmetic mean of function f defined by*

$$F(x,y)=\begin{cases}\dfrac{1}{x-y}\int_x^y f(t)\,\mathrm{d}t, & x\ne y,\\ f(x), & x=y.\end{cases}$$

Then

(*i*) *function $F(x,y)$ is increasing (decreasing) on I^2 if and only if f is an increasing (decreasing) function on I,*

(*ii*) *$F(x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if $f(x)$ is convex (concave) on I.*

Lemma 5. *Let $a\le b, u(t)=tb+(1-t)a, v(t)=ta+(1-t)b$. If $1/2\le t_2\le t_1\le 1$ or $0\le t_1\le t_2\le 1$, then*

$$(u(t_2),v(t_2))\prec(u(t_1),v(t_1))\prec(a,b). \tag{2}$$

Proof. Case 1. When $1/2 \le t_2 \le t_1 \le 1$, it is easy to see that $u(t_1) \ge v(t_1), u(t_2) \ge v(t_2), u(t_1) \ge u(t_2)$ and $u(t_2) + v(t_2) = u(t_1) + v(t_1) = a + b$, this is (1) holds.

Case 2. When $0 \le t_1 \le t_2 \le 1$, then $1/2 \le 1 - t_2 \le 1 - t_1 \le 1$, by the Case 1, it follows

$$(u(1-t_2), v(1-t_2)) \prec (u(1-t_1), v(1-t_1)),$$

i.e. $(u(t_2), v(t_2)) \prec (u(t_1), v(t_1))$. □

Lemma 6 ([?])**.** *Let $l, t, p, q \in \mathbb{R}_{++}, p > q$ and $p + q \le 3(l + t)$. Assume also that $1/3 \le l/t \le 3$ or $q \le l + t$. Then*

$$G(l, t; x, y) \le (p/q)^{1/(p-q)} E(p, q; x, y).$$

Lemma 7. *Let*

$$g(t, z) = \frac{z^t + 1}{t(z^{t-1} - 1)}.$$

Then for fixed $z > 1$,

(*i*) *$g(t, z)$ is increasing on $(-\infty, 0)$ with t;*
(*ii*) *$g(t, z)$ is increasing on $(0, \xi_z)$ with t,*
(*iii*) *$g(t)$ is decreasing on $(\xi_z, 1)$ or $(1, +\infty)$ with t,*

where ξ_z is an zero of the function

$$g_1(t, z) = t(z^t + z^{t-1}) \ln z + (z^t + 1)(z^{t-1} - 1)$$

with $0 < \xi_z < 1/2$.

Proof. Differentiate $g(t, z)$ with respect to t to obtain

$$\frac{\partial g(t,z)}{\partial t} = \frac{tz^t(z^{t-1} - 1)\ln z - (z^t + 1)(z^{t-1} - 1) - tz^{t-1}(z^t + 1)\ln z}{t^2(z^{t-1} - 1)^2} = -\frac{g_1(t, z)}{t^2(z^{t-1} - 1)^2}.$$

For fixed $z > 1$, $g_1(t, z) < 0$ and $\frac{\partial g(t,z)}{\partial t} > 0$ on $(-\infty, 0)$, then $g(t, z)$ increases on $(-\infty, 0)$ with t, and $g_1(t, z) > 0$ and $\frac{\partial g(t,z)}{\partial t} < 0$ on $(1, +\infty)$, then $g(t, z)$ decreases on $(1, +\infty)$ with t.

Differentiate $g_1(t, z)$ with respect to t to obtain

$$\frac{\partial g_1(t,z)}{\partial t} = [2z^{2t-1} + 2z^{t-1} + t(z^t + z^{t-1})\ln z] \ln z.$$

Since $\frac{\partial g_1(t,z)}{\partial t} > 0$ on $(0, 1)$, $g_1(t, z)$ increases on $(0, 1)$, it following that $g_1(0, z) \le g_1(t, z) \le g_1(1, z)$. Furthermore, $g_1(0, z) = 2(z^{-1} - 1) < 0$ and $g_1(1, z) = (z + 1)\ln z > 0$, hence there exist $\xi_z \in (0, 1)$ such that $g_1(\xi_z, z) = 0$, and $g_1(t, z) \le 0$ and $\frac{\partial g(t,z)}{\partial t} \ge 0$ for $0 < t \le \xi_z$, and $g_1(t, z) > 0$ and $\frac{\partial g(t,z)}{\partial t} < 0$ for $\xi_z < t < 1$. this is, $g(t, z)$ increases on $(0, \xi_z)$ and decreases on $(\xi_z, 1)$.

Differentiate $g_1(t, z)$ with respect to z to obtain

$$\begin{aligned}\frac{\partial g_1(t,z)}{\partial z} &= tz^{t-1}(z^{t-1} - 1) + (t - 1)z^{t-2}(z^t + 1) \\ &\quad + t(z^{t-1} + z^{t-2}) + t[tz^{t-1} + (t - 1)z^{t-2}]\ln z \\ &= (2t - 1)z^{2t-2} + t^2 z^{t-1}\ln z + (2t - 1)z^{t-2} + (t^2 - t)z^{t-2}\ln z.\end{aligned}$$

For $1 > t \geq 1/2$, we have

$$\begin{aligned}\frac{\partial g_1(t,z)}{\partial z} &\geq t^2 z^{t-1}\ln z + (2t-1)z^{t-2} + (t^2-t)z^{t-2}\ln z \\ &= (t^2 z + t^2 - t)z^{t-2}\ln z \\ &> (2t^2 - t)z^{t-2}\ln z = t(2t-1)z^{t-2}\ln z \geq 0.\end{aligned}$$

Hence, for $1 > t \geq 1/2$, $g_1(t,z)$ increases on $(1,+\infty)$ with z, and then

$$g_1(t,z) > \lim_{z\to 1^+} g_1(t,z) = g_1(t,1) = 0.$$

Thus we conclude that $0 < \xi_z < 1/2$. □

Lemma 8. *For fixed (x,y) with $x > y > 0$, If $(r,s) \in \{r > 1, s < 0, r+s \leq 1\} \cup \{1 < r \leq s\} \cup \{0 < r \leq 1-r \leq s < 1\} \cup \{1/2 \leq r \leq s < 1\}$, then*

$$s(x^r + y^r)(x^{s-1} - y^{s-1}) \geq r(x^s + y^s)(x^{r-1} - y^{r-1}), \tag{3}$$

if $(r,s) \in \{s > 1, r < 0, r+s \leq 1\} \cup \{r \leq s < 0\}$, then (2) is reversed.

Proof. Let $g(t) = \frac{z^t+1}{t(z^{t-1}-1)}$ with $z = x/y > 1$. Notice that $y > 0$, it is easy to see that (2) equivalent to $g(r) \geq g(s)$. For $r > 1$, we first prove that $g(r) \geq g(1-r)$, i.e.

$$\frac{y(z^r+1)}{r(z^{r-1}-1)} \geq \frac{y(z^{1-r}+1)}{(1-r)(z^{-r}-1)} = \frac{y(z^r+z)}{(r-1)(z^r-1)}.$$

It is sufficient prove that

$$h(z) := (r-1)(z^r-1)(z^r+1) - r(z^{r-1}-1)(z^r+z) \geq 0.$$

Directly calculating yields

$$h(z) = (r-1)z^{2r} - rx^{2r-1} + rx - r + 1,$$

$$h'(z) = 2r(r-1)z^{2r-1} - r(2r-1)z^{2r-2} + r,$$

$$h''(z) = 2r(r-1)(2r-1)z^{2r-3}(z-1).$$

By $r > 1$, and $z > 1$, it follows $h''(z) > 0$. therefore, $h'(z) > h'(1) = 0$, moreover, $h(z) > h(1) = 0$, i.e. $g(r) \geq g(1-r)$.

If $r > 1, s < 0, r+s \leq 1$, then $s \leq 1-r < 0$, from (i) of Lemma 7, we have $g(r) \geq g(s)$, i.e. (2) holds.

If $s > 1, r < 0, r+s \leq 1$, replacing r by s and replacing s by r in the above case, it follows that $g(r) \leq g(s)$, i.e. (2) is reversed.

If $0 < r \leq 1/2 \leq 1-r \leq s < 1$, then $h''(z) > 0$, it follows $h'(z) > h'(1) = 0$, moreover, $h(z) > h(1) = 0$, i.e. $g(r) \geq g(1-r)$, from (iii) of Lemma 7, we have $g(r) \geq g(1-r) \geq g(s)$, i.e. (2) holds.

If $1/2 \leq r \leq s < 1$ or $1 < r \leq s$, from (iii) of Lemma 7, we have $g(r) \geq g(s)$ i.e. (2) holds.

If $r \leq s < 0$, from (i) of Lemma 7, we have $g(r) \leq g(s)$ i.e. (2) is reversed. □

3. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

Theorem 1. *For fixed $(x,y)\in\mathbb{R}^2_{++}$,*

(*i*) *$G(r,s;x,y)$ is increasing with $(r,s)\in\mathbb{R}^2$,*
(*ii*) *$G(r,s;x,y)$ is Schur-concave with $(r,s)\in\mathbb{R}^2_{+}$,*
(*iii*) *$G(r,s;x,y)$ is Schur-convex with $(r,s)\in\mathbb{R}^2_{-}$.*

Proof. Let $g(t)=\dfrac{x^t\ln x+y^t\ln y}{x^t+y^t}$. It is easy to check that

$$\ln G(r,s;x,y)=\frac{1}{r-s}\int_r^s g(t)\ dt,\quad r\neq s,$$

(see the proof of Lemma 3.1 of in [?]). By computing, we get

$$g^{'}(t)=\frac{x^ty^t}{(x^t+y^t)^2}(\ln x-\ln y)^2,$$

$$g^{''}(t)=-\frac{x^ty^t(\ln x-\ln y)^2}{(x^t-y^t)^4}(\ln x-\ln y)(x^{2t}-y^{2t}).$$

(*i*) Since $g^{'}(t)\geq 0$, $g(t)$ is increasing in $\mathbb{R}$. By (*i*) in lemma 4 , it follows that $\ln G(r,s;x,y)$ is increasing in $\mathbb{R}^2$ with (r,s), and so does $G(r,s;x,y)$.

(*ii*) When $(r,s)\in\mathbb{R}^2_+$, $\ln u$ and u^{2t} are increasing in $\mathbb{R}_+$ with u, hence $(\ln x-\ln y)(x^{2t}-y^{2t})\geq 0$, moreover $g^{''}(t)\leq 0$. That is, $g(t)$ is concave in $\mathbb{R}_+$. By (*ii*) in lemma 4 , it follows that

$$F(r,s)=\begin{cases}\ln G(r,s;x,y), & r\neq s,\\ g(t), & r=s\end{cases}$$

is Schur-concave in $\mathbb{R}^2_+$ with (r,s), and so does $G(r,s;x,y)$ by Corollary 6.14 in [11, p. 64].

(*iii*) When $(r,s)\in\mathbb{R}^2_-$, $\ln u$ is increasing and u^{2t} is decreasing in $\mathbb{R}_+$ with u, hence $(\ln x-\ln y)(x^{2t}-y^{2t})\leq 0$, moreover, $g^{''}(t)\geq 0$. That is, $g(t)$ is convex in $\mathbb{R}_-$. By (*ii*) in lemma 4 , it follows that $F(r,s)$ is Schur-convex in $\mathbb{R}^2_-$ with (r,s), and so does $G(r,s;x,y)$.

The proof is complete. □

Remark 1. How about the Schur-convexity of the function $G(r,s;x,y)$ on the set $\{r>0,s<0\}$ or $\{r<0,s>0\}$ with (r,s)? This can not be confirmed.

Theorem 2. *For fixed $(r,s)\in\mathbb{R}^2$,*

(*i*) *if $(r,s)\in\{r\geq 0,s\geq 0,r+s\geq 1\}$, then $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_{++}$;*
(*ii*) *if $(r,s)\in\{r\leq 0,r+s\leq 1\}\cup\{s\leq 0,r+s\leq 1\}$, then $G(r,s;x,y)$ is the Schur-concave with $(x,y)\in\mathbb{R}^2_{++}$.*

Proof. Let $\varphi(x,y)=\dfrac{x^s+y^s}{x^r+y^r}$. When $r\neq s$, for fixed $(x,y)\in\mathbb{R}^2$, we have

$$\frac{\partial\varphi}{\partial x}=\frac{sx^{s-1}(x^r+y^r)-rx^{r-1}(x^s+y^s)}{(x^r+y^r)^2},$$

$$\frac{\partial\varphi}{\partial y}=\frac{sy^{s-1}(x^r+y^r)-ry^{r-1}(x^s+y^s)}{(x^r+y^r)^2}.$$

$$\begin{aligned}\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}&=\frac{s(x^r+y^r)(x^{s-1}-y^{s-1})-r(x^s+y^s)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2}\\&=\frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot\frac{(r-1)(x^{s-1}-y^{s-1})}{(s-1)(x^{r-1}-y^{r-1})}-\frac{r}{s}\cdot\frac{x^s+y^s}{x^r+y^r}\right]\\&=\frac{s(x^{r-1}-y^{r-1})}{(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right],\end{aligned}$$

and then

$$\Delta:=(x-y)\left(\frac{\partial G}{\partial x}-\frac{\partial G}{\partial y}\right)=\frac{x-y}{s-r}\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)$$
$$=\frac{s(x-y)(x^{r-1}-y^{r-1})}{(s-r)(x^r+y^r)}\left[\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right]\varphi^{\frac{1}{s-r}-1}(x,y)$$

In lemma 6, taking $l=r,t=s,p=r-1,q=s-1$, we have

$$\begin{cases}l>0,t>0,p>0,q>0\\p>q\\p+q\le 3(l+t)\\1/3\le l/t\le 3\end{cases}\Leftrightarrow\begin{cases}r>1,s>1\\r>s\\r+s\ge -1\\s/3\le r\le 3s\end{cases}\Leftrightarrow 3s\ge r>s>1$$

and

$$\begin{cases}l>0,t>0,p>0,q>0\\p>q\\p+q\le 3(l+t)\\q\le l+t\end{cases}\Leftrightarrow\begin{cases}r>1,s>1\\r>s\\r+s\ge -1\\r\ge -1\end{cases}\Leftrightarrow r>s>1.$$

Hence, when $r>s>1$, we have

$$G(r,s;x,y)\le\left(\frac{r-1}{s-1}\right)^{\frac{1}{r-s}}E(r-1,s-1;x,y),$$

i.e.

$$G^{s-r}(r,s;x,y)\ge\frac{s-1}{r-1}\cdot E^{s-r}(r-1,s-1;x,y). \tag{4}$$

When $r>s>1$, we have $s-r<0$ and $(x-y)(x^{r-1}-y^{r-1})\ge 0$. Combining with (3), it follows that $\Delta\ge 0$. By lemma 2, $G(r,s;x,y)$ is the Schur-convex with $(x,y)\in\mathbb{R}^2_{++}$.

Now we consider other cases. Notice that

$$(x-y)\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)=\frac{s(x^r+y^r)(x-y)(x^{s-1}-y^{s-1})-r(x^s+y^s)(x-y)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2},$$

when $r\ge 1, 0\le s\le 1$, since t^{r-1} and t^{s-1} is increasing and decreasing in $\mathbb{R}_{++}$ respectively, it follows that $(x-y)(x^{s-1}-y^{s-1})\ge 0$ and $(x-y)(x^{r-1}-y^{r-1})\le 0$, moreover, $(x-y)\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\le 0$ and

$$\Delta=\frac{x-y}{s-r}\left(\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)\ge 0.$$

That is, when $r \geq 1, 0 \leq s \leq 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y) \in \mathbb{R}^2_{++}$.

When $r < 0, 0 < s \leq 1$, since t^{r-1} and t^{s-1} are decreasing in $\mathbb{R}_{++}$, it follows that $(x-y)(x^{s-1}-y^{s-1}) \leq 0$ and $(x-y)(x^{r-1}-y^{r-1}) \leq 0$, moreover, $(x-y)\left(\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y}\right) \leq 0$ and $\Delta \leq 0$, that is, when $r < 0, 0 < s \leq 1$, $G(r,s;x,y)$ is the Schur-concave with $(x,y) \in \mathbb{R}^2_{++}$.

Without loss of generality,we may assume $x > y > 0$. Notice that

$$\Delta = \frac{x-y}{s-r} \cdot \frac{s(x^r+y^r)(x^{s-1}-y^{s-1}) - r(x^s+y^s)(x^{r-1}-y^{r-1})}{(x^r+y^r)^2} \varphi^{\frac{1}{s-r}-1}(x,y),$$

When $r > 1, s < 0, r+s \leq 1$, from Lemma 8, it following that $\Delta \leq 0$, i.e. $G(r,s;x,y)$ is the Schur-concave with $(x,y) \in \mathbb{R}^2_{++}$.

Similarly, we can prove that when $r \leq s < 0$, $G(r,s;x,y)$ is the Schur-concave with $(x,y) \in \mathbb{R}^2_{++}$, and when $0 < r \leq 1-r \leq s$ or $1/2 \leq r \leq s < 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y) \in \mathbb{R}^2_{++}$.

When $r = s \geq 1$, let

$$\psi(x,y) = \frac{x^s \ln x + y^s \ln y}{x^r + y^r} = \frac{x^s \ln x + y^s \ln y}{x^s + y^s}.$$

Then

$$\frac{\partial \psi}{\partial x} = \frac{x^{s-1} h(x,y)}{(x^s+y^s)^2}, \quad \frac{\partial \psi}{\partial y} = \frac{y^{s-1} k(x,y)}{(x^s+y^s)^2},$$

where

$$h(x,y) = (s \ln x + 1)(x^s + y^s) - s(x^s \ln x + y^s \ln y),$$
$$k(x,y) = (s \ln y + 1)(x^s + y^s) - s(x^s \ln x + y^s \ln y).$$

By computing,

$$\begin{aligned} & x^{s-1}h(x,y) - y^{s-1}k(x,y) \\ &= (x^s+y^s)\left[x^{s-1}(s\ln x + 1) - y^{s-1}(s \ln y + 1)\right] - s(x^s \ln x + y^s \ln y)(x^{s-1} - y^{s-1}) \\ &= s^{s-1}y^{s-1}(x+y)(\ln x - \ln y) + (x^{s-1}-y^{s-1})(x^s+y^s), \end{aligned}$$

and then,

$$\begin{aligned} & (x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) = (x-y)\left(\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y}\right) e^{\psi(x,y)} \\ &= \frac{sx^{s-1}y^{s-1}(x+y)(x-y)(\ln x - \ln y) + (x-y)(x^{s-1}-y^{s-1})(x^s+y^s)}{(x^s+y^s)^2} e^{\psi(x,y)}. \end{aligned}$$

Since $\ln t$ and t^{s-1} are increasing in $\mathbb{R}_+$ with t for $s \geq 1$, therefore, $(x-y)(\ln x - \ln y) \geq 0$ and $(x-y)(x^{s-1}-y^{s-1}) \geq 0$, moreover, $(x-y)\left(\frac{\partial G}{\partial x} - \frac{\partial G}{\partial y}\right) \geq 0$. That is, when $r = s \geq 1$, $G(r,s;x,y)$ is the Schur-convex with $(x,y) \in \mathbb{R}^2_{++}$.

In conclusion, if $(r,s) \in \{r > s > 1\} \cup \{r = s \geq 1\} \cup \{r \geq 1, 0 \leq s \leq 1\} \cup \{0 < r \leq 1-r \leq s\} \cup \{1/2 \leq r \leq s < 1\}$, then $G(r,s;x,y)$ is the Schur-convex with $(x,y) \in \mathbb{R}^2_{++}$, and if $(r,s) \in \{r < 0, 0 < s \leq 1\} \cup \{r > 1, s < 0, r+s \leq 1\} \cup \{r \leq s < 0\}$, then $G(r,s;x,y)$ is the Schur-concave with $(x,y) \in \mathbb{R}^2_{++}$.

Since $G(r,s;x,y)$ is symmetric with (r,s), if $(r,s) \in \{s > r > 1\} \cup \{s \geq 1, 0 \leq r \leq 1\} \cup \{0 < s \leq 1-s \leq r\} \cup \{1/2 \leq s \leq r < 1\}$, then $G(r,s;x,y)$ is also the Schur-convex with $(x,y) \in \mathbb{R}^2_{++}$, and if $(r,s) \in \{s < 0, 0 < r \leq 1\} \cup \{s > 1, r < 0, r+s \leq 1\} \cup \{s \leq r < 0\}$, then $G(r,s;x,y)$ is also the Schur-concave with $(x,y) \in \mathbb{R}^2_{++}$.

The proof is complete. □

Remark 2. The Schur-convexity of the function $G(r,s;x,y)$ on the set $\{s<0, r+s>1\}$ or $\{r<0, r+s>1\}$ or $\{r>0, s>0, r+s<1\}$ with (x,y) is uncertainty.

Example 1. Let $(r,s)=(2.5,-1.2)$. It is clear that $(2.5,-1.2)\in\{s<0, r+s>1\}$. For $(3,3)\prec(5,1)$, directly calculating yields

$$G(2.5,-1.2;3,3)=3.000000000>G(2.5,-1.2;5,1)=2.873884533.$$

But, for$(1.25,1.25)\prec(1.5,1)$, directly calculating yields

$$G(2.5,-1.2;1.25,1.25)=1.25.0000000<G(2.5,-1.2;1.5,1)=1.256253447.$$

Example 2. Let $(r,s)=(-0.2,1.5)$. It is clear that $(-0.2,1.5)\in\{r<0, r+s>1\}$. For $(8,8)\prec(15,1)$, directly calculating yields

$$G(-0.2,1.5;8,8)=8.000000000<G(-0.2,1.5;15,1)=8.412747770.$$

But, for$(25.5,25.5)\prec(50,1)$, directly calculating yields

$$G(-0.2,1.5;25.5,25.5)=25.5.0000000>G(-0.2,1.5;50,1)=25.32833093.$$

Example 3. Let $(r,s)=(0.6,0.2)$. It is clear that $(0.6,0.2)\in\{r>0, s>0, r+s<1\}$. For $(10.5,10.5)\prec(20.9,0.1)$, directly calculating yields

$$G(0.6,0.2;10.5,10.5)=10.5.0000000<G(0.6,0.2;20.9,0.1)=11.03249418.$$

But, for$(10.5,10.5)\prec(18,3)$, directly calculating yields

$$G(0.6,0.2;10.5,10.5)=10.50000000>G(0.6,0.2;18,3)=9.970045812.$$

Theorem 3. *If $(r,s)\in\mathbb{R}^2_{++}$, then $G(r,s;x,y)$ is the Schur-geometrically convex with $(x,y)\in\mathbb{R}^2_{++}$.*

Proof. Let $\varphi(x,y)=\dfrac{x^s+y^s}{x^r+y^r}$. When $r\neq s$, for fixed $(x,y)\in\mathbb{R}^2$, we have

$$x\frac{\partial\varphi}{\partial x}=\frac{sx^s(x^r+y^r)-rx^r(x^s+y^s)}{(x^r+y^r)^2},$$

$$y\frac{\partial\varphi}{\partial y}=\frac{sy^s(x^r+y^r)-ry^r(x^s+y^s)}{(x^r+y^r)^2}.$$

$$\begin{aligned}x\frac{\partial\varphi}{\partial x}-y\frac{\partial\varphi}{\partial y}&=\frac{s(x^r+y^r)(x^s-y^s)-r(x^s+y^s)(x^r-y^r)}{(x^r+y^r)^2}\\&=\frac{s(x^r-y^r)}{x^r+y^r}\left[\frac{s}{r}\cdot\frac{r(x^s-y^s)}{s(x^r-y^r)}-\frac{r}{s}\cdot\frac{x^s+y^s}{x^r+y^r}\right]\\&=\frac{s(x^r-y^r)}{x^r+y^r}\left[\frac{s}{r}\cdot E^{s-r}(r,s;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right],\end{aligned}$$

and then

$$\begin{aligned}&(\ln x-\ln y)\left(x\frac{\partial G}{\partial x}-y\frac{\partial G}{\partial y}\right)=\frac{\ln x-\ln y}{s-r}\left(x\frac{\partial\varphi}{\partial x}-y\frac{\partial\varphi}{\partial y}\right)\varphi^{\frac{1}{s-r}-1}(x,y)\\&=\frac{s(\ln x-\ln y)(x^r-y^r)}{(s-r)(x^r+y^r)}\left[\frac{s}{r}\cdot E^{s-r}(r,s;x,y)-\frac{r}{s}\cdot G^{s-r}(r,s;x,y)\right]\varphi^{\frac{1}{s-r}-1}(x,y)\end{aligned}$$

In Lemma 6, taking $l = p = r, t = q = s$, we have

$$\begin{cases} l > 0, t > 0, p > 0, q > 0 \\ p > q \\ p + q \le 3(l + t) \\ 1/3 \le l/t \le 3 \end{cases} \Leftrightarrow \begin{cases} r > 0, s > 0 \\ r > s \\ r + s \ge -1 \\ s/3 \le r \le 3s \end{cases} \Leftrightarrow 3s \ge r > s > 0$$

and

$$\begin{cases} l > 0, t > 0, p > 0, q > 0 \\ p > q \\ p + q \le 3(l + t) \\ q \le l + t \end{cases} \Leftrightarrow \begin{cases} r > 0, s > 0 \\ r > s \\ r + s \ge -1 \\ r \ge 0 \end{cases} \Leftrightarrow r > s > 0.$$

Hence, when $r > s > 0$, we have

$$G(r, s; x, y) \le \left(\frac{r}{s}\right)^{\frac{1}{r-s}} E(r, s; x, y),$$

i.e.

$$G^{s-r}(r, s; x, y) \ge \frac{s}{r} \cdot E^{s-r}(r, s; x, y). \tag{5}$$

When $r > s > 0$, we have $s - r < 0$, and since $\ln t$ and t^r are increasing in $\mathbb{R}_+$ with t, therefore $(\ln x - \ln y)(x^r - y^r) \ge 0$. Combining with (4), it follows that $(\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) \ge 0$. By Lemma 3, $G(r, s; x, y)$ is the Schur-geometrically convex with (x, y) in $\mathbb{R}^2_{++}$. Since $G(r, s; x, y)$ is symmetric with (r, s), when $s > r > 0$, $G(r, s; x, y)$ is also the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_{++}$.

Now we consider other cases.

Without loss of generality,we may assume $x > y > 0$. Notice that

$$\Lambda = \frac{\ln x - \ln y}{s - r} \cdot \frac{s(x^r + y^r)(x^s - y^s) - r(x^s + y^s)(x^r - y^r)}{(x^r + y^r)^2} \varphi^{\frac{1}{s-r}-1}(x, y),$$

when $r = s > 0$, we have

$$\frac{\partial \psi}{\partial x} = \frac{x^{s-1}h(x, y)}{(x^s + y^s)^2}, \quad \frac{\partial \psi}{\partial y} = \frac{x^{s-1}k(x, y)}{(x^s + y^s)^2},$$

where $h(x, y), k(x, y)$ and $\psi(x, y)$ are same as in Theorem 2.

By computing,

$$x^s h(x, y) - y^s k(x, y) = s^s y^s (x + y)(\ln x - \ln y) + (x^s - y^s)(x^s + y^s),$$

and then,

$$\begin{aligned} (\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) &= (\ln x - \ln y)\left(x\frac{\partial \psi}{\partial x} - y\frac{\partial \psi}{\partial y}\right) e^{\psi(x,y)} \\ = \frac{s x^s y^s (x + y)(\ln x - \ln y)^2 + (\ln x - \ln y)(x^s - y^s)(x^s + y^s)}{(x^s + y^s)^2} e^{\psi(x,y)}. \end{aligned}$$

Since when $s > 0$, $\ln t$ and t^s are increasing in $\mathbb{R}_+$, $(\ln x - \ln y)(x^s - y^s) \ge 0$, moreover, $(\ln x - \ln y)\left(x\frac{\partial G}{\partial x} - y\frac{\partial G}{\partial y}\right) \ge 0$. That is, when $r = s > 0$, $G(r, s; x, y)$ is the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_{++}$.

In conclusion, if $(r, s) \in \{r > s > 0\} \cup \{s > r > 0\} \cup \{r = s > 0\} = \mathbb{R}^2_{++}$, $G(r, s; x, y)$ is the Schur-geometrically convex with $(x, y) \in \mathbb{R}^2_{++}$.

The proof is complete.

□

4. Applications

Theorem 4. *Let $u(t) = ts + (1-t)r, v(t) = tr + (1-t)s$, and let $\frac{1}{2} \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1$. If $s \ge r \ge 0$, then for fixed $(x, y) \in \mathbb{R}^2_{++}$, we have*

$$G\left(\frac{r+s}{2}, \frac{r+s}{2}; x, y\right) \le G\left(u(t_2), v(t_2); x, y\right)$$
$$\le G\left(u(t_1), v(t_1); x, y\right) \le G(r, s; x, y) \le G(r+s, 0; x, y), \tag{6}$$

and if $s \le r \le 0$, then inequalities in (5) *are all reversed.*

Proof. From lemma 5, we have

$$\left(\frac{r+s}{2}, \frac{r+s}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (r, s).$$

and it is clear that $(r, s) \prec (r+s, 0)$, by Theorem 1, it follows that (5) are holds.

The proof is complete. □

Theorem 5. *Let $(x, y) \in \mathbb{R}^2_{++}$, $u(t) = ty + (1-t)x, v(t) = tx + (1-t)y$, and let $\frac{1}{2} \le t_2 \le t_1 \le 1$ or $0 \le t_1 \le t_2 \le 1$. If $(r, s) \in \{r \ge 0, s \ge 0, r+s \ge 1\} \subseteq \mathbb{R}^2$, then for fixed $(r, s) \in \mathbb{R}^2$, we have*

$$G\left(r, s; \frac{x+y}{2}, \frac{x+y}{2}\right) \le G\left(r, s; u(t_2), v(t_2)\right)$$
$$\le G\left(r, s; u(t_1), v(t_1)\right) \le G(r, s; x, y) \le G(r, s; x+y, 0). \tag{7}$$

if $(r, s) \in \{r \le 0, r+s \le 1\} \cup \{s \le 0, r+s \le 1\} \subseteq \mathbb{R}^2$, then inequalities in (6) *are all reversed.*

Proof. From lemma 5, we have

$$\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (r, s).$$

and it is clear that $(x, y) \prec (x+y-\varepsilon, \varepsilon)$, where ε is enough small positive number.

If $(r, s) \in \{r \ge 1, s > 0\} \cup \{0 < r < 1, s \ge 1\}$, by Theorem 1, and let $\varepsilon \to 0$, it follows that (6) are holds. If $(r, s) \in \{r < 0, 0 < s < 1\} \cup \{0 < r < 1, s < 0\}$, then inequalities in (6) are all reversed.

The proof is complete. □

Theorem 6. *Let $(x, y) \in \mathbb{R}^2_{++}$. For fixed $(r, s) \in \mathbb{R}^2_{++}$, we have*

$$G\left(r, s; \sqrt{xy}, \sqrt{xy}\right) \le G(r, s; x, y). \tag{8}$$

Proof. Since $(\ln\sqrt{xy}, \ln\sqrt{xy}) \prec (\ln x, \ln y)$, by Theorem 3, it follows that (7) is holds.

The proof is complete. □

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