An inequality between the arithmetic mean of some numbers and the arithmetic mean of their images through a convex function

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Abstract

The purpose of this note is to present a relation between the arithmetic mean, of a finite number of real numbers, and the arithmetic mean of their images through a convex function. Some applications of this inequality are also included.

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1 Introduction

Let $n \geq 1$ be a fixed natural number, and I an interval of real numbers. For every $a = (a_1, a_2, \ldots, a_n) \in I^n$, the *arithmetic mean* associated to a is defined as:

$$A_n[a] \quad := \quad \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Let $I \subset \mathbf{R}$ be an interval. If $f : I \to \mathbf{R}$ is a convex (concave) function, then the well known Jensen inequality (see [1]–[3], [7], [8]) says that:

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \quad \leq (\geq) \quad \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n},$$

which can be written, using the above notation, as:

$$f\left(A_n[a]\right) \quad \leq (\geq) \quad A_n\left[f(a)\right]. \tag{1.1}$$

The inequality (1.1) represents a relation between the image of $A_n[a]$, through f, and the arithmetic mean of the images of the numbers $\{a_i\}_{1 \leq i \leq n}$, i.e., $A_n[f(a)]$.

We are interested in finding a direct relation between $A_n[a]$ and $A_n[f(a)]$.

2 Main result

We present now the main result of this note.

Proposition 2.1 *If*

1. $f : I \subset \mathbf{R} \to \mathbf{R}$ *is a convex (concave) function;*
2. $a_i \in I \cap [m,\, M]$, *for all* $i \in \{1,\, 2,\, \ldots,\, n\}$, *where* $m,\, M \in \mathbf{R}$,

then the following inequality:

$$(M-m)\cdot A_n\left[f(a)\right]-\left[f(M)-f(m)\right]\cdot A_n[a] \;\le (\ge)\; M\cdot f(m)-m\cdot f(M) \tag{2.2}$$

holds.

Proof. Let $i \in \{1,\, 2,\, \ldots,\, n\}$ be fixed. Since $a_i \in [m,\, M]$, a_i can be written as the following convex combination of M and m:

$$a_i \;=\; \frac{a_i-m}{M-m}\cdot M+\frac{M-a_i}{M-m}\cdot m.$$

If we denote $p_i := (a_i-m)/(M-m)$ and $q_i := (M-a_i)/(M-m)$, then since $p_i \ge 0$, $q_i \ge 0$, and $p_i+q_i = 1$, we conclude from the definition of convexity of f, that:

$$\begin{aligned} f(a_i) &= f(p_i\cdot M+q_i\cdot m)\\ &\le p_i f(M)+q_i f(m)\\ &= \frac{(a_i-m)f(M)+(M-a_i)f(m)}{M-m}. \end{aligned} \tag{2.3}$$

Summing up in the last inequality from $i=1$ to $i=n$, we obtain:

$$\begin{aligned} \sum_{i=1}^{n} f(a_i) &\le \frac{\left(\sum_{i=1}^{n} a_i-n\cdot m\right)\cdot f(M)+\left(n\cdot M-\sum_{i=1}^{n} a_i\right)\cdot f(m)}{M-m}\\ &= \frac{n\cdot\left(\frac{1}{n}\sum_{i=1}^{n} a_i-m\right)\cdot f(M)+n\cdot\left(M-\frac{1}{n}\cdot\sum_{i=1}^{n} a_i\right)\cdot f(m)}{M-m} \end{aligned}$$

After multiplying both sides of the last relation by $(M-m)/n$, we obtain:

$$(M-m)\cdot A_n\left[f(a)\right] \;\le\; (A_n[a]-m)\cdot f(M)+(M-A_n[a])\cdot f(m),$$

which is equivalent to:

$$(M-m)\cdot A_n\left[f(a)\right]-\left[f(M)-f(m)\right]\cdot A_n[a] \;\le\; M\cdot f(m)-m\cdot f(M).$$

If f is concave all the inequalities from this proof are reversed. □

Corollary 2.2 *If we impose the additional condition that* $f(m) < f(M)$ *to the assumptions from Proposition 2.1, then:*

$$\frac{A_n\left[f(a)\right]}{f(M)-f(m)}-\frac{A_n[a]}{M-m} \;\le (\ge)\; \frac{M\cdot f(m)-m\cdot f(M)}{(M-m)\cdot(f(M)-f(m))}. \tag{2.4}$$

Proof. Since $f(m) \neq f(M)$, we have $m \neq M$. Dividing both sides of the inequality (2.2) by the strictly positive number $(M-m)\,(f(M)-f(m))$, we obtain (2.4). $\square$

Application 2.3 *Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = e^x$. It is clear that f is convex and strictly increasing on $\mathbf{R}$. Thus for all $\{a_i\}_{i=1}^n \subset [m,\, M] \subset \mathbf{R}$, such that $m < M$, we conclude from (2.4), that:*

$$\frac{e^{a_1}+e^{a_2}+\cdots+e^{a_n}}{n(e^M-e^n)} - \frac{a_1+a_2+\cdots+a_n}{n(M-m)} \quad \leq \quad \frac{Me^m-me^M}{(M-m)(e^M-e^m)}. \tag{2.5}$$

If $a_1 \leq a_2 \leq \cdots \leq a_n$, with at least one of these inequalities being strict, then setting: $m = a_1$ and $M = a_n$, we can rewrite (2.5) as:

$$\frac{e^{a_1}+e^{a_2}+\cdots+e^{a_n}}{n(e^{a_n}-e^{a_1})} - \frac{a_1+a_2+\cdots+a_n}{n(a_n-a_1)} \quad \leq \quad \frac{a_n e^{a_1}-a_1 e^{a_n}}{(a_n-a_1)(e^{a_n}-e^{a_1})}. \tag{2.6}$$

Corollary 2.4 *If $a_i \in [m,\, \infty)$, $m,\, M \in (0,\, \infty)$, for all $i \in \{1,\, 2,\, \ldots,\, n\}$, then:*

$$G_n^{M-m}[a] \quad \geq \quad M^{A_n[a]-m} \cdot m^{M-A_n[a]},$$

where $G_n[a] := \sqrt[n]{a_1 \cdot a_2 \cdots a_n}$ is the geometric mean of the numbers $\{a_i\}_{i=1}^n$.

Proof. Since the function $f : (0,\, \infty) \to \mathbf{R}$, $f(x) = \ln x$ is concave, it follows from the inequality (2.2) that:

$$\begin{aligned}
& (M-m)\cdot A_n[\ln a] - (\ln M - \ln m)\cdot A_n[a] \geq M\ln m - m\ln M \\
\Leftrightarrow\;\; & (M-m)\cdot \frac{\ln a_1 + \ln a_2 + \cdots + \ln a_n}{n} - \ln\left(\frac{M}{m}\right)\cdot A_n[a] \geq \ln\left(\frac{m^M}{M^m}\right) \\
\Leftrightarrow\;\; & \ln G_n^{M-m}[a] - \ln\left[\left(\frac{M}{m}\right)^{A_n[a]}\right] \geq \ln\left(\frac{m^M}{M^m}\right) \\
\Leftrightarrow\;\; & \ln\left(\frac{G_n^{M-m}[a]}{\left(\frac{M}{m}\right)^{A_n[a]}}\right) \geq \ln\left(\frac{m^M}{M^m}\right) \\
\Leftrightarrow\;\; & G_n^{M-m}[a] \geq \left(\frac{M}{m}\right)^{A_n[a]} \cdot \frac{m^M}{M^m} \\
\Leftrightarrow\;\; & G_n^{M-m}[a] \geq M^{A_n[a]-m} \cdot m^{M-A_n[a]}.
\end{aligned}$$

$\square$

If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then we can take $m = a_1$ and $M = a_n$, and obtain:

$$G_n^{a_n-a_1}[a] \quad \geq \quad a_n^{A_n[a]-a_1} \cdot a_1^{a_n-A_n[a]}. \tag{2.7}$$

This inequality holds even when $a_1 = a_2 = \cdots = a_n$.

Application 2.5 *Let* $a_i = i + 1$, *for all* $i \in \{1, 2, \ldots, n\}$. *Then we can take* $m = a_1 = 2$ *and* $M = a_n = n + 1$. *We have:*

$$\begin{aligned} A_n[a] &= \frac{2 + 3 + \cdots + (n+1)}{n} \\ &= \frac{n+3}{2} \\ &= \frac{n}{2} + \frac{3}{2}. \end{aligned}$$

It follows now from (2.7) that:

$$\left[\sqrt[n]{(n+1)!}\right]^{n-1} \;\geq\; 2^{\frac{n-1}{2}} \cdot (n+1)^{\frac{n-1}{2}}.$$

This is equivalent to:

$$\sqrt[n]{(n+1)!} \;\geq\; \sqrt{2(n+1)}.$$

Corollary 2.6 *If* $a_i \in [m, M]$, *for all* $i \in \{1, 2, \ldots, n\}$, *and* $[m, M] \subset (0, \infty)$, *then:*

$$A_n[a] \;\leq\; M + m - \frac{M \cdot m}{H_n[a]}, \tag{2.8}$$

where

$$H_n[a] \;:=\; \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}$$

is the harmonic mean of the numbers $\{a_i\}_{i=1}^{n}$.

Proof. The function $f : (0, \infty) \to \mathbf{R}$, $f(x) = 1/x$ is convex. Applying the inequality (2.2) to this function, we get:

$$\begin{aligned} & (M - m) \cdot A_n[1/a] - \left(\frac{1}{M} - \frac{1}{m}\right) \cdot A_n[a] \leq M \cdot \frac{1}{m} - m \cdot \frac{1}{M} \\ \Leftrightarrow\; & (M - m) \cdot \frac{1/a_1 + 1/a_2 + \cdots + 1/a_n}{n} - \frac{m - M}{M \cdot m} \cdot A_n[a] \leq \frac{M}{m} - \frac{m}{M} \\ \Leftrightarrow\; & (M - m) \cdot \frac{1}{H_n[a]} + \frac{M - m}{M \cdot m} \cdot A_n[a] \leq \frac{M^2 - m^2}{M \cdot m} \\ \Leftrightarrow\; & \frac{1}{H_n[a]} + \frac{A_n[a]}{M \cdot m} \leq \frac{M + m}{M \cdot m} \\ \Leftrightarrow\; & A_n[a] \leq M + m - \frac{M \cdot m}{H_n[a]}. \end{aligned}$$

□

If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, then by taking $m := a_1$ and $M := a_n$ in (2.8), we obtain:

$$A_n[a] \;\leq\; a_1 + a_n - \frac{a_1 \cdot a_n}{H_n[a]}.$$

Alternative characterizations for means, obtained by different methods, can be found in [4]–[6].

Observation 2.7 *The means $A_n[a]$ and $H_n[a]$ make sense for negative numbers, too. If $\{a_i\}_{i=1}^n \subset (-\infty,\, 0)$, then the inequality (2.8) is reversed, which can be proved very simply by multiplying both sides of this inequality by -1.*

Application 2.8 *Let $a_i = i$, for all $i \in \{1,\, 2,\, \dots,\, n\}$. Then we can take $m = a_1 = 1$, $M = a_n = n$, and we have $A_n[a] = (n+1)/2$. It follows now from (2.8) that:*

$$\frac{n+1}{2} \quad \leq \quad n + 1 - \frac{n \cdot 1}{H_n[a]}.$$

This is equivalent to

$$\frac{n}{H_n[a]} \quad \leq \quad \frac{n+1}{2}.$$

The last inequality means:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \leq \quad \frac{n+1}{2}.$$

Corollary 2.9 *Let $r \geq 1$ and $\{a_i\}_{i=1}^n \subset [m,\, M] \subset (0,\, \infty)$. Then the following inequality holds:*

$$(M - m) \cdot (M_n^r[a])^r - (M^r - m^r) \cdot A_n[a] \quad \leq \quad M \cdot m^r - m \cdot M^r, \tag{2.9}$$

where:

$$M_n^r[a] \quad := \quad \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n} \right)^{1/r}$$

is the power mean of order r *(called also the* Hölder mean of order r*) of the numbers a_1, a_2, $\dots$, a_n. In particular if* $\min\{a_i \mid 1 \leq i \leq n\} = a_1$, $\max\{a_i \mid 1 \leq i \leq n\} = a_n$, *and $a_1 < a_n$, then:*

$$\frac{(M_n^r[a])^r}{a_n^r - a_1^r} - \frac{A_n[a]}{a_n - a_1} \quad \leq \quad \frac{a_n \cdot a_1^r - a_1 \cdot a_n^r}{(a_n - a_1) \cdot (a_n^r - a_1^r)}. \tag{2.10}$$

Proof. Everything follow by applying the inequality (2.2) to the convex and increasing function $f : [0,\, \infty) \to \mathbf{R}$, $f(x) = x^r$. □

Observation 2.10 *All the inequalities from this paper can be reformulated using weighted means, too.*

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