ON GENERALIZATIONS OF THE HADAMARD INEQUALITY FOR (α, m)-CONVEX FUNCTIONS

ERHAN SET*, M. SARDARI, M. EMIN ÖZDEMIR, AND J. ROOIN

ABSTRACT. In this paper we establish several **H**adamard type integral inequalities for (α, m) −convex functions.

1. INTRODUCTION

One of the most important integral inequalities for convex functions is Hadamard's inequality (or Hermite-Hadamard integral inequality).The following double inequality is well known as Hadamard Integral Inequality in the literature.

Theorem 1. *If f is convex function on $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Proof. See [1]. □

If the function f is concave, (1.1) can be written as following:

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

Some further properties of the Hadamard integral inequality are given in [8], [9].

In the literature, the concepts of $m-$convexity and (α, m) −convexity are well known.The concept of $m-$convexity was first introduced by G. Toader in [11] (see also [4], [5]) and it is defined as follows:

The function $f : [0, b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.2}$$

The class of (α, m) −convex functions was also first introduced in [7], and it is defined as follows:

The function $f : [0, b] \to \mathbb{R},\ b > 0$, is said to be (α, m) −convex , where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^{\alpha}f(x) + m(1-t^{\alpha})f(y) \tag{1.3}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

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*Corresponding author.

It can be easily seen that for $(\alpha, m) \in \{(0,0), (1,1)\ (1,m)\}$ one obtains the following classes of functions: increasing, convex and $m-$convex functions respectively.The interested reader can find more about partial ordering of convexity in [10, P. 8,280].We refer to interested reader for many papers connected with $m-$convex and $(\alpha, m)-$convex functions [2],[3],[5]. There are similar inequalities for $s-$convex functions in [6].

The following theorems were proved by S.S. Dragomir and C.E.M. Pearce, in [4]:

Theorem 2. *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.4}$$

Some generalizations of this result can be found in [2],[3].

Theorem 3. *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a,b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}{}_{a}^{b}\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ &\leq \frac{1}{4}\left[\left(\frac{f(a)+f(b)}{2}\right)+m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)\right. \\ &\quad \left.+m^2\left(\frac{f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)}{2}\right)\right] \end{aligned} \tag{1.5}$$

Theorem 4. *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $f \in L_1[am,b]$ where $0 \leq a < b$, then one has the inequality:*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^b f(x)dx\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.6}$$

The goal of this paper is to obtain the generalizations of Theorems 1,2,3,4 for $(\alpha, m)-$convex functions.

2. Inequalities For (α, m)-Convex functions

The following theorem is a generalization of Hadamard Inequality.

Theorem 5. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha, m)-$convex function with $0 \leq a < b$ and $(\alpha, m) \in [0,1] \times (0,1]$. If $f \in L_1\left[m^2a, (2-m)b\right] \cap L_1\left[ma, \frac{(2-m)b}{m}\right]$, then one has* *the inequalities:*

$$
\begin{aligned}
& f\left(\frac{2-m}{2}b+\frac{m}{2}(ma)\right) \\
& \leq \frac{1}{2^{\alpha}}\frac{1}{b(2-m)-m^{2}a}\left\{\int_{m^{2}a}^{(2-m)b}\left[f(x)+\left(2^{\alpha}-1\right)m\right.\right. \\
& \left.\left.\times f\left(\frac{(2-m)b}{m}\left(1-\frac{x-m^{2}a}{2b-mb-m^{2}a}\right)+m\frac{x-m^{2}a}{2b-mb-m^{2}a}a\right)\right]dx\right\} \qquad (2.1)\\
& \leq \frac{1}{2^{\alpha}(\alpha+1)}\left[f((2-m)b)\right. \\
& \left.+\left(\alpha+\left(2^{\alpha}-1\right)\right)mf(am)+\left(2^{\alpha}-1\right)\alpha m^{2}f\left(\frac{(2-m)b}{m^{2}}\right)\right]
\end{aligned}
$$

Proof. Let $U_1 = t(2-m)b + (1-t)m^2a$ and $U_2 = (1-t)(2-m)b + tm^2a$, where $t \in [0,1]$ is arbitrary. Then we get

$$
f\left(\frac{U_1+U_2}{2}\right) = f\left(\frac{2-m}{2}b+\frac{m}{2}(ma)\right).
$$

By the (α, m) −convexity of f we can write the following inequality:

$$
\begin{aligned}
f\left(\frac{2-m}{2}b+\frac{m}{2}(ma)\right) &= f\left(\frac{U_1+U_2}{2}\right) \\
&\leq \frac{1}{2^{\alpha}}f(U_1)+\left(1-\frac{1}{2^{\alpha}}\right)mf\left(\frac{U_2}{m}\right) \\
&= \frac{1}{2^{\alpha}}\left[f(U_1)+\left(2^{\alpha}-1\right)mf\left(\frac{U_2}{m}\right)\right],
\end{aligned}
$$

or

$$
\begin{aligned}
f\left(\frac{2-m}{2}b+\frac{m}{2}(ma)\right) &\leq \frac{1}{2^{\alpha}}\left[f\left(t(2-m)b+(1-t)m^{2}a\right)\right. \\
&\left.+\left(2^{\alpha}-1\right)mf\left(\frac{(1-t)(2-m)b}{m}+tma\right)\right]
\end{aligned}
$$

Integrating over $t \in [0,1]$, we get

$$
\begin{aligned}
& f\left(\frac{2-m}{2}b+\frac{m}{2}(ma)\right) \\
& \leq \frac{1}{2^{\alpha}}\int_0^1 \left[f\left(t(2-m)b+(1-t)m^2a\right)\right. \\
& \left. +\left(2^{\alpha}-1\right)mf\left(\frac{(1-t)(2-m)b}{m}+tma\right)\right]dt \qquad (2.2) \\
& =\frac{1}{2^{\alpha}}\frac{1}{b(2-m)-m^2a}\left\{\int_{m^2a}^{(2-m)b}\left[f(x)+\left(2^{\alpha}-1\right)m\right.\right. \\
& \left.\left.\times f\left(\frac{(2-m)b}{m}\left(1-\frac{x-m^2a}{2b-mb-m^2a}\right)+m\frac{x-m^2a}{2b-mb-m^2a}a\right)\right]dx\right\},
\end{aligned}
$$

where we used the change of the variable $x=t(2-m)b+(1-t)m^2a$, or $t=\frac{x-m^2a}{2b-mb-m^2a}$, and so

$$
\int_0^1 f\left(t(2-m)b+(1-t)m^2a\right)dt=\frac{1}{(2-m)b-m^2a}\int_{m^2a}^{(2-m)b} f(x)\,dx,
$$

and

$$
\begin{aligned}
& \int_0^1 f\left(\frac{(1-t)(2-m)b}{m}+tma\right)dt \\
& =\frac{1}{(2-m)b-m^2a} \\
& \times\int_{m^2a}^{(2-m)b} f\left(\frac{(2-m)b}{m}\left(1-\frac{x-m^2a}{2b-mb-m^2a}\right)+m\frac{x-m^2a}{2b-mb-m^2a}a\right)dx.
\end{aligned}
$$

This completes the proof of the first inequality in (2.1).

Next, by the $(\alpha,m)-$convexity of f, we also have

$$
\begin{aligned}
f\left(t(2-m)b+(1-t)m^2a\right) &= f\left(t(2-m)b+m(1-t)ma\right) \\
&\leq t^{\alpha}f\left((2-m)b\right)+m\left(1-t^{\alpha}\right)f(ma)
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\frac{(1-t)(2-m)b}{m}+tma\right) &= f\left(t(ma)+m(1-t)\left(\frac{(2-m)b}{m^2}\right)\right) \\
&\leq t^{\alpha}f(ma)+m\left(1-t^{\alpha}\right)f\left(\frac{(2-m)b}{m^2}\right).
\end{aligned}
$$

So,
(2.3)

$$\frac{1}{2^{\alpha}}\left[f\left(t\left(2-m\right)b+\left(1-t\right)m^{2}a\right)+\left(2^{\alpha}-1\right)mf\left(\frac{\left(1-t\right)\left(2-m\right)b}{m}+tma\right)\right]$$

$$\leq\frac{1}{2^{\alpha}}\left\{t^{\alpha}f\left(\left(2-m\right)b\right)+m\left(1-t^{\alpha}\right)f\left(ma\right)\right.$$

$$\left.+\left(2^{\alpha}-1\right)m\left[t^{\alpha}f\left(ma\right)+m\left(1-t^{\alpha}\right)f\left(\frac{\left(2-m\right)b}{m^{2}}\right)\right]\right\}.$$

Integrating (2.3) over t on $[0,1]$, we get

$$\frac{1}{2^{\alpha}}\frac{1}{\left(2-m\right)b-m^{2}a}\left\{\int_{m^{2}a}^{(2-m)b}\left[f\left(x\right)+\left(2^{\alpha}-1\right)m\right.\right.$$

$$\left.\left.\times f\left(\frac{\left(2-m\right)b}{m}\left(1-\frac{x-m^{2}a}{2b-mb-m^{2}a}\right)+m\frac{x-m^{2}a}{2b-mb-m^{2}a}a\right)\right]dx\right\}$$

$$\leq\frac{1}{2^{\alpha}\left(\alpha+1\right)}\left\{f\left(\left(2-m\right)b\right)\right.$$

$$\left.+\left(\alpha+\left(2^{\alpha}-1\right)\right)mf\left(am\right)+\left(2^{\alpha}-1\right)\alpha m^{2}f\left(\frac{\left(2-m\right)b}{m^{2}}\right)\right\}.$$

This completes the proof of the second inequality in (2.1). □

Remark 1. *Choosing* $(\alpha,m)=(1,1)$ *in (2.1), from the first and the second inequalities of (2.1), respectively, we obtain*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\frac{1}{b-a}\left[\int_a^b\left[f(x)+f(a+b-x)\right]dx\right] \\ &= \frac{1}{2}\frac{1}{b-a}\left[\int_a^b f(x)dx+\int_a^b f(x)dx\right] \\ &= \frac{1}{b-a}\int_a^b f(x)dx \end{aligned}$$

and

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq \frac{1}{4}\left[f(b)+2f(a)+f(b)\right] \\ &= \frac{f(a)+f(b)}{2} \end{aligned}$$

Note that, we used

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx,$$

and so

$$\int_a^b\left[f(x)+f(a+b-x)\right]dx=2\int_a^b f(x)dx.$$

Clearly, we can drop the assumption $f \in L_1\left[m^2a,(2-m)b\right] \cap L_1\left[ma, \frac{(2-m)b}{m}\right] = L_1[a,b]$, *and (2.1) exactly becomes the Hermite-Hadamard inequalities for* $(\alpha,m) = (1,1)$.

Theorem 6. *Let* $f:[0,\infty)\to\mathbb{R}$ *be an* $(\alpha,m)-$*convex function with* $(\alpha,m)\in(0,1]^2$. *If* $0\le a<b<\infty$ *and* $f\in L_1[a,b]$, *then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}, \frac{f(b)+\alpha m f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \tag{2.4}$$

Proof. Since f is $(\alpha,m)-$convex, we have

$$f(tx+m(1-t)y)\le t^\alpha f(x)+m(1-t^\alpha)f(y)$$

for all $x,y\ge 0$, which gives:

$$f(ta+(1-t)b)\le t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a)\le t^\alpha f(b)+m(1-t^\alpha)f\left(\frac{a}{m}\right)$$

for all $t\in[0,1]$. Integrating on $[0,1]$, we obtain

$$\int_0^1 f(ta+(1-t)b)\,dt \le \frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}$$

and

$$\int_0^1 f(tb+(1-t)a)\,dt \le \frac{f(b)+\alpha m f\left(\frac{a}{m}\right)}{\alpha+1}.$$

However,

$$\int_0^1 f(ta+(1-t)b)\,dt = \int_0^1 f(tb+(1-t)a)\,dt = \frac{1}{b-a}\int_a^b f(x)dx$$

and the inequality (2.4) is obtained. □

Remark 2. *The inequality (2.4) yields inequality (1.4) for* $\alpha=1$.

Theorem 7. *Let* $f:[0,\infty)\to\mathbb{R}$ *be an* $(\alpha,m)-$*convex function with* $(\alpha,m)\in(0,1]^2$. *If* $0\le a<b<\infty$ *and* $f\in L_1[a,b]\cap L_1\left[\frac{a}{m},\frac{b}{m}\right]$, *then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2^\alpha(b-a)}\int_a^b \left[f(x)+m(2^\alpha-1)f\left(\frac{x}{m}\right)\right]dx \\ &\le \frac{1}{2^{\alpha+1}(\alpha+1)}\left[(f(a)+f(b))\right. \\ &+ m(\alpha+2^\alpha-1)\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right) \\ &+ \left.\alpha m^2(2^\alpha-1)\left(f\left(\frac{a}{m^2}\right)+f\left(\frac{b}{m^2}\right)\right)\right] \end{aligned} \tag{2.5}$$

Proof. By the (α,m) −convexity of f, we have

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &= f\left(\frac{x}{2}+m\frac{y}{2m}\right) \\ &\leq \frac{1}{2^{\alpha}}f(x)+m\left(1-\frac{1}{2^{\alpha}}\right)f\left(\frac{y}{m}\right) \\ &= \frac{1}{2^{\alpha}}\left[f(x)-mf\left(\frac{y}{m}\right)\right]+mf\left(\frac{y}{m}\right) \end{aligned}$$

for all $x,y\in[0,\infty)$.

Now, if we choose $x=ta+(1-t)b$ and $y=(1-t)a+tb$, we deduce

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^{\alpha}}\left[f(ta+(1-t)b)-mf\left(\frac{(1-t)a+tb}{m}\right)\right]+mf\left(\frac{(1-t)a+tb}{m}\right) \\ &= \frac{1}{2^{\alpha}}\left[f(ta+(1-t)b)+m(2^{\alpha}-1)f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right] \end{aligned}$$

for all $t\in[0,1]$.

Integrating over $t\in[0,1]$, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^{\alpha}}\left[\int_0^1 f(ta+(1-t)b)\,dt\right. \\ &\quad \left.+m(2^{\alpha}-1)\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt\right]. \end{aligned} \tag{2.6}$$

Taking into account that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt=\frac{1}{b-a}\int_a^b f(\frac{x}{m})dx,$$

we deduce from (2.6) the first inequality in (2.5).

Next, by the (α,m) −convexity of f, we also have

$$\begin{aligned} &\frac{1}{2^{\alpha}}\left[f(ta+(1-t)b)+m(2^{\alpha}-1)f\left(t\frac{b}{m}+(1-t)\frac{a}{m}\right)\right] \\ \leq\; &\frac{1}{2^{\alpha}}\left[t^{\alpha}f(a)+m(1-t^{\alpha})f\left(\frac{b}{m}\right)\right. \\ &\left.+m(2^{\alpha}-1)\left(t^{\alpha}f\left(\frac{b}{m}\right)+m(1-t^{\alpha})f\left(\frac{a}{m^2}\right)\right)\right]. \end{aligned} \tag{2.7}$$

Integrating over t on $[0,1]$, we get

$$\begin{aligned}
(2.8)\qquad & \frac{1}{2^{\alpha}(b-a)}\int_a^b \left(f(x)+m(2^{\alpha}-1)f(\frac{x}{m})\right)dx \\
\leq\ & \frac{1}{2^{\alpha}}\left[f(a)\int_0^1 t^{\alpha}dt+mf\left(\frac{b}{m}\right)\int_0^1(1-t^{\alpha})\,dt+\right. \\
& +m(2^{\alpha}-1)f\left(\frac{b}{m}\right)\int_0^1 t^{\alpha}dt \\
& \left.+m^2(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\int_0^1(1-t^{\alpha})\,dt\right] \\
=\ & \frac{1}{2^{\alpha}(\alpha+1)}\left[f(a)+m(\alpha+2^{\alpha}-1)f\left(\frac{b}{m}\right)+\alpha m^2(2^{\alpha}-1)f\left(\frac{a}{m^2}\right)\right].
\end{aligned}$$

Similarly, changing the roles of a and b, we get

$$\begin{aligned}
(2.9)\qquad & \frac{1}{2^{\alpha}(b-a)}\int_a^b \left(f(x)+m(2^{\alpha}-1)f(\frac{x}{m})\right)dx \\
\leq\ & \frac{1}{2^{\alpha}(\alpha+1)}\left[f(b)+m(\alpha+2^{\alpha}-1)f\left(\frac{a}{m}\right)+\alpha m^2(2^{\alpha}-1)f\left(\frac{b}{m^2}\right)\right].
\end{aligned}$$

Now adding (2.8) and (2.9) with each other, we obtain the second inequality in (2.5). □

Remark 3. *Choosing $\alpha=1$ in (2.5), we get (1.5).*

Remark 4. *The inequality (2.5) yields Hadamard inequality (1.1) for $\alpha=1$ and $m=1$.*

Theorem 8. *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$(2.10)\qquad \frac{1}{b-a}\int_a^b f(x)dx\leq\frac{1}{2}\left[\frac{f(a)+f(b)+\alpha m\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{\alpha+1}\right].$$

Proof. By the $(\alpha,m)-$convexity of f, we can write

$$f(ta+(1-t)b)\leq t^{\alpha}f(a)+m(1-t^{\alpha})f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a)\leq t^{\alpha}f(b)+m(1-t^{\alpha})f\left(\frac{a}{m}\right)$$

for all $t\in[0,1]$.

Adding the above inequalities, we get

$$f(ta+(1-t)b)+f(tb+(1-t)a)\leq t^{\alpha}f(a)+m(1-t^{\alpha})f\left(\frac{b}{m}\right)+t^{\alpha}f(b)+m(1-t^{\alpha})f\left(\frac{a}{m}\right).$$

Integrating over $t \in [0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f(ta+(1-t)b)\,dt + \int_0^1 f(tb+(1-t)a)\,dt \qquad (2.11)\\ \leq\ & \int_0^1 t^{\alpha}(f(a)+f(b))\,dt + \int_0^1 m(1-t^{\alpha})\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)dt \\ =\ & \frac{f(a)+f(b)}{\alpha+1} + \frac{m\alpha}{\alpha+1}\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right) \\ =\ & \frac{f(a)+f(b)+m\alpha\left(f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right)}{\alpha+1}. \end{aligned}$$

As it is easy to see that

$$\int_0^1 f(ta+(1-t)b)\,dt = \int_0^1 f(tb+(1-t)a)\,dt = \frac{1}{b-a}\int_a^b f(x)dx,$$

from (2.11) we deduce the desired result, namely, the inequality (2.10). □

Remark 5. *The inequality (2.10) yields the right side of Hadamard inequality (1.1) for $\alpha = 1$ and $m = 1$.*

Theorem 9. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m) \in (0,1]^2$. If $L_1[am,b]$ where $0 \leq a < b$, then one has the inequalities:*

$$\begin{aligned} &\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx \qquad (2.12)\\ \leq\ & \frac{1}{(\alpha+1)}\left[(f(a)+f(b))(1+m\alpha)\right]. \end{aligned}$$

Proof. By $(\alpha,m)-$convexity of f, for all $t \in [0,1]$, we can write:

$$\begin{aligned} f(ta+m(1-t)b) &\leq t^{\alpha}f(a)+m(1-t^{\alpha})f(b), \\ f(tb+m(1-t)a) &\leq t^{\alpha}f(b)+m(1-t^{\alpha})f(a), \\ f((1-t)a+mtb) &\leq (1-t)^{\alpha}f(a)+m(1-(1-t)^{\alpha})f(b), \\ f((1-t)b+mta) &\leq (1-t)^{\alpha}f(b)+m(1-(1-t)^{\alpha})f(a). \end{aligned}$$

Adding the above inequalities with each other, we get:

$$\begin{aligned} & f(ta+m(1-t)b)+f(tb+m(1-t)a) \\ & +f((1-t)a+mtb)+f((1-t)b+mta) \\ & \leq [t^{\alpha}+m(1-t^{\alpha})+(1-t)^{\alpha}+m(1-(1-t)^{\alpha})](f(a)+f(b)). \end{aligned}$$

Now integrating over $t \in [0,1]$ and taking into account that:

$$\int_0^1 f(ta+m(1-t)b)\,dt = \int_0^1 f((1-t)a+mtb)\,dt = \frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f(tb+m(1-t)a)\,dt = \int_0^1 f((1-t)b+mta)\,dt = \frac{1}{b-ma}\int_{ma}^{b} f(x)dx,$$

we obtain the inequality (2.12). □

Remark 6. *Choosing $\alpha = 1$ in (2.12), we obtain (1.6).*

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Department of Mathematics, Faculty of Education K.K., Atatürk University, 25240, Erzurum, Turkey
E-mail address: erhanset@yahoo.com

Institute for Advanced Studies in Basic Sciences, P.O.Box 45195-1159 Zanjan, IRAN.
E-mail address: m_sardari@iasbs.ac.ir

Graduate School of Natural and Applied Sciences, Ağri İbrahim Çeçen University, Ağri, Turkey
E-mail address: emos@atauni.edu.tr

Institute for Advanced Studies in Basic Sciences, P.O.Box 45195-1159 Zanjan, IRAN.
E-mail address: rooin@iasbs.ac.ir