NEW INEQUALITIES OF HERMITE-HADAMARD'S TYPE

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Abstract. In this paper, we establish several inequalities for some differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Some applications for special means of real numbers are also provided.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [5]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

Definition 1. *The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality, (see [1],[2], [3] and [4]) which has generated a wide range of directions for extension and a rich mathematical literature.

In [1] in order to prove some inequalities related to Hadamard's inequality Dragomir and Agarwal used the following lemma.

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta + (1-t)b)dt.$$

In [1], Dragomir and Agarwal established the following results connected with the right part of (1.1) as well as to apply them for some elementary inequalities for real numbers and numerical integration.

2000 *Mathematics Subject Classification.* 26D15,

Key words and phrases. convex function, Hermite-Hadamard inequality.

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Theorem 1. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$,and $f' \in L(a, b)$. If the mapping $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left(\frac{|f'(a)|+|f'(b)|}{8}\right).$$

Theorem 2. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}.$$

In [2], using the same Lemma 1, Pearce and Pečarić proved the following theorem.

Theorem 3. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard's type. Finally, we gave some applications for special means of real numbers.

2. Main Results

We will establish some new results connected with the right-hand side of (1.1) used the following Lemma. Now, we give the following new Lemma for our results:

Lemma 2. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned}(2.1)\qquad &\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{(b-a)}{2}\int_0^1\int_0^1 \left(f'(ta+(1-t)b)-f'(sa+(1-s)b)\right)(s-t)\,dtds.\end{aligned}$$

Proof. By integration by parts, we get

$$\int_0^1\int_0^1 \left(f'(ta+(1-t)b)-f'(sa+(1-s)b)\right)(s-t)\,dtds$$

$$=\int_0^1 \left\{\int_0^1 f'(ta+(1-t)b)(s-t)\,dt - \int_0^1 f'(sa+(1-s)b)(s-t)\,dt\right\} ds$$

$$=\int_0^1 \left\{(s-t)\left.\frac{f(ta+(1-t)b)}{a-b}\right|_0^1 + \int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt + \left(\frac{1}{2}-s\right) f'(sa+(1-s)b)\right\} ds$$

$$=\int_0^1 \left\{(s-1)\frac{f(a)}{a-b}-s\frac{f(b)}{a-b}+\int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt+\left(\frac{1}{2}-s\right)f'(sa+(1-s)b)\right\}ds$$

$$=\left.\frac{(s-1)^2}{2}\frac{f(a)}{a-b}-\frac{s^2}{2}\frac{f(b)}{a-b}\right|_0^1+\int_0^1 \frac{f(ta+(1-t)b)}{a-b}dt$$

$$+\left.\left(\frac{1}{2}-s\right)\frac{f(sa+(1-s)b)}{a-b}\right|_0^1+\int_0^1 \frac{f(sa+(1-s)b)}{a-b}ds$$

$$=\frac{f(a)+f(b)}{b-a}-\frac{2}{b-a}\int_0^1 f(ta+(1-t)b)dt.$$

Using the change of the variable $x=ta+(1-t)b$ for $t\in[0,1]$, and multiplying the both sides by $\frac{b-a}{2}$, we obtain (2.1), which completes the proof. □

Theorem 4. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\left(\frac{|f'(a)|+|f'(b)|}{6}\right).$$

Proof. From Lemma 2 and the convexity of $|f'|$, it follows that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{(b-a)}{2}\left[\int_0^1\int_0^1|f'(ta+(1-t)b)-f'(sa+(1-s)b)|\,|s-t|\,dtds\right]$$

$$\leq\frac{(b-a)}{2}\left[\int_0^1\int_0^1|f'(ta+(1-t)b)|\,|s-t|\,dtds+\int_0^1\int_0^1|f'(sa+(1-s)b)|\,|s-t|\,dtds\right]$$

$$\leq(b-a)\int_0^1\int_0^1\left(t\,|s-t|\,|f'(a)|+(1-t)\,|s-t|\,|f'(b)|\right)dtds$$

$$=(b-a)\left(\frac{|f'(a)|+|f'(b)|}{6}\right)$$

where we have used the facts that

$$\int_0^1\int_0^1 t\,|s-t|\,dtds=\int_0^1\int_0^1(1-t)\,|s-t|\,dtds=\frac{1}{6}$$

which completes the proof. □

Theorem 5. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, using the well known Hölder inequality for double integrals, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)}{2}\left[\int_0^1\int_0^1 |f'(ta+(1-t)b)-f'(sa+(1-s)b)|\,|s-t|\,dtds\right] \\
&\leq \frac{(b-a)}{2}\left[\int_0^1\int_0^1 |f'(ta+(1-t)b)|\,|s-t|\,dtds+\int_0^1\int_0^1 |f'(sa+(1-s)b)|\,|s-t|\,dtds\right] \\
&\leq (b-a)\left[\left(\int_0^1\int_0^1 |f'(ta+(1-t)b)|^q\,dtds\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1 |s-t|^p\,dtds\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f'(ta+(1-t)b)|^q \leq t\,|f'(a)|^q+(1-t)\,|f'(b)|^q,$$

hence

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq (b-a)\left(\int_0^1\int_0^1 |s-t|^p\,dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1 \left(t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right)dtds\right)^{\frac{1}{q}} \\
&= (b-a)\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, and we have used the fact that

$$\int_0^1\int_0^1 |s-t|^p\,dtds=\int_0^1\left[\int_0^s (s-t)^p\,dt+\int_s^1 (t-s)^p\,dt\right]ds=\frac{2}{(p+1)(p+2)}$$

which completes the proof. □

Corollary 1. *Under the assumptations of Theorem 5 with $q=p=2$, then we have the following inequality,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{\sqrt{6}}\left(\frac{|f'(a)|^2+|f'(b)|^2}{2}\right)^{\frac{1}{2}}. \tag{2.2}$$

Proof. From Lemma 2, using the Cauchy-Shawartz integral inequality and the convexity of $|f'|^2$, we obtain (2.2). □

Theorem 6. *Let $f: I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|^q$ is convex on $[a,b]$, $q\geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{3}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. From Lemma 2, using the well known power mean inequality for double integrals, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{2}\left[\int_0^1\int_0^1 \left|f'(ta+(1-t)b)-f'(sa+(1-s)b)\right|\left|s-t\right| dtds\right]$$

$$\leq \frac{(b-a)}{2}\left[\int_0^1\int_0^1 \left|f'(ta+(1-t)b)\right|\left|s-t\right| dtds+\int_0^1\int_0^1 \left|f'(sa+(1-s)b)\right|\left|s-t\right| dtds\right]$$

$$\leq (b-a)\left[\left(\int_0^1\int_0^1 \left|s-t\right| dtds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 \left|s-t\right|\left|f'(ta+(1-t)b)\right|^q dtds\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$\left|f'(ta+(1-t)b)\right|^q \leq t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q,$$

hence

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(\int_0^1\int_0^1 \left|s-t\right| dtds\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 \left|s-t\right|\left(t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right) dtds\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)}{3}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

where we have used the facts that

$$\int_0^1\int_0^1 \left|s-t\right| dtds=\frac{1}{3}$$

and

$$\int_0^1\int_0^1 t\left|s-t\right| dtds=\int_0^1\int_0^1 (1-t)\left|s-t\right| dtds=\frac{1}{6}$$

which completes the proof. □

3. Applications to Some Special Means

We now consider the applications of our Theorems to the following special means:

(a) The arithmetic mean: $A=A(a,b):=\dfrac{a+b}{2},\ \ a,b\geq 0,$

(b) The geometric mean: $G=G(a,b):=\sqrt{ab},\ \ a,b\geq 0,$

(c) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\ \ a,b\geq 0,$$

(d) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & if\quad a=b \\ \frac{b-a}{\ln b-\ln a} & if\quad a\neq b \end{cases},\qquad a,b>0,$$

(e) The Identric mean:

$$I=I\left(a,b\right):=\begin{cases} a & \text{if} \quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a\neq b \end{cases}, \quad a,b>0,$$

(f) The $p-$logarithmic mean

$$L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a\neq b \\ a & \text{if} \quad a=b \end{cases}, \quad p\in\mathbb{R}\diagdown\{-1,0\}\,;\ a,b>0.$$

The following proposition holds:

Proposition 1. *Let $a,b\in\mathbb{R}$, $0\notin[a,b]$, and $n\in\mathbb{Z}$ and $|n|\geq 2$. Then, for all $q>1$,we have*

$$\left|A\left(a^n,b^n\right)-L_n^n\left(a,b\right)\right|\leq|n|\,\frac{(b-a)}{3}A\left(|a|^{(n-1)},|b|^{(n-1)}\right).$$

Proof. The proof is immediate from Theorem 4 applied for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{Z}$ and $|n|\geq 2$. □

Proposition 2. *Let $a,b\in\mathbb{R}$, $0\notin[a,b]$, and $n\in\mathbb{Z}$, $|n|\geq 2$. Then, for all $q>1$,we have*

$$\left|A\left(a^n,b^n\right)-L_n^n\left(a,b\right)\right|\leq|n|\left(b-a\right)\left(\frac{2}{\left(p+1\right)\left(p+2\right)}\right)^{\frac{1}{p}}\left[A\left(|a|^{q(n-1)},|b|^{q(n-1)}\right)\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied for $f(x)=x^n$, $x\in\mathbb{R}$, $n\in\mathbb{Z}$ and $|n|\geq 2$. □

Proposition 3. *Let $a,b\in[0,\infty)$ and $a<b$. Then, for all $q>1$, we have*

$$\ln\left[\frac{I\left(a+1,b+1\right)}{G\left(a+1,b+1\right)}\right]\leq\frac{(b-a)}{(b+1)\left(a+1\right)}\left(\frac{2}{\left(p+1\right)\left(p+2\right)}\right)^{\frac{1}{p}}\left[A\left((b+1)^q,(a+1)^q\right)\right]^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied to $f:[0,\infty)\rightarrow(-\infty,0]$, $f(x)=-\ln\left(x+1\right)$ and the details are omitted. □

Proposition 4. *Let $a,b\in\mathbb{R}$, $0\notin[a,b]$, then, for all $q\geq 1$,the following inequality holds:*

$$\left|H^{-1}\left(a,b\right)-L^{-1}\left(a,b\right)\right|\leq\frac{b-a}{3}\left[A\left(|a|^{-2q},|b|^{-2q}\right)\right]^{\frac{1}{q}}$$

Proof. The proof is obvious from Theorem 6 applied for $f(x)=\frac{1}{x}$, $x\in[a,b]$. □

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