Solutions of Two Conjectures on Inequalities with Power-exponential Functions

Yin Li*

Abstract: In this paper, we mainly prove two conjectures posted by V. Cîrtoaje. Related problems are also presented.

Key Words: Bernoulli's inequality; Monotonicity; Concave function; Power-exponential Functions

2000 AMS. Sub. Class: 26D10

Email: *yinli*_79@163.com

1. Introduction

In 2006, A. Zeikii posted and proved on the Mathlinks Forum the following inequality

$$a^a + b^b \geq a^b + b^a \tag{1.1}$$

where a and b are positive real numbers less than or equal to 1. In addition, he conjectured that the following inequality holds under the same conditions

$$a^{2a} + b^{2b} \geq a^{2b} + b^{2a}. \tag{1.2}$$

Later, V. Cîrtoaje proposed and proved the open inequality

$$a^{ea} + b^{eb} \geq a^{eb} + b^{ea} \tag{1.3}$$

for either $a \geq b \geq \frac{1}{e}$ or $\frac{1}{e} \geq a \geq b > 0$ in [1]. In the same paper, he also posted several conjectures.

Recently, L. Matejíčka proved Conjecture 4.6 posted by V. Cîrtoaje in [2].

In this paper, we mainly prove Conjecture 4.7 and Conjecture 4.8. In addition, we give a partial solution to Conjecture 4.3. Finally, other related open problems are presented.

2. Main Results

Conjecture 4.7. If a and b are nonnegative real numbers such that $a + b = 2$, then

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2. \tag{2.1}$$

*Department of Mathematics and Information Science, Binzhou University,Shandong 256600

Proof. Without loss of generality, assume that $a \geq b$.
Case $0.3 \leq b \leq 1$. Since $0.7 \leq a-1 \leq 1$, by Bernoulli's inequality we have

$$a^b \leq 1 + b(a-1) = 1 + b + b^2$$

and

$$b^a = bb^{a-1} \leq b[1 + (a-1)(b-1)] = b^2(2-b).$$

Therefore

$$\begin{aligned} & a^{3b} + b^{3a} + (\frac{a-b}{2})^4 - 2 \\ & \leqslant (1+b+b^2)^3 + b^6(2-b)^3 + (1-b)^4 - 2 \cdot \\ & \leqslant -b(b-1)^3(b^5 - 3b^4 + 3b^2 + 3b - 1) \leqslant 0 \end{aligned}$$

(In fact, we denote $g(b) = b^5 - 3b^4 + 3b^2 + 3b - 1$. Since $g(0) = -1 < 0$ and $g(0.3) = 0.14813 > 0$, then $g(b) = 0$ only exist a real root $\xi \in (0, 0.3)$ by theorem of zero point and simple argument.)
Case $0 \leq b < \frac{1}{9}$. Since

$$a^{3b} \leqslant (1+b)^3 = 1 + 3b + 3b^2 + b^3$$

and $b^{3a} \leqslant 8b^6$, we have

$$\begin{aligned} & a^{3b} + b^{3a} + (\frac{a-b}{2})^4 - 2 \\ & \leqslant -b + 9b^2 - 3b^3 + b^4 + 8b^6 \leq 0 \end{aligned}.$$

Case $\frac{1}{9} \leq b < 0.3$. Defining function

$$f(b) = (2-b)^{3b} + b^{3(2-b)} + (1-b)^4 - 2,$$

we have

$$f(b) \leqslant -b(b-1)^3 g(b)$$

where $g(b) = b^5 - 3b^4 + 3b^2 + 3b - 1$. [See Case $0.3 \leq b \leq 1$.]

Owing to $g(\xi) = 0$, equality $f(b) = 0$ don't have real root on $[\frac{1}{9}, \xi] \subset [\frac{1}{9}, 0.3]$. Hence $f(b) \leq f(\frac{1}{9}) < 0$ or $f(b) \leq f(\xi) \leq f(0.3) < 0$. This complete the proof.
Conjecture 4.8. If a and b are nonnegative real numbers such that $a + b = 1$, then

$$a^{2b} + b^{2a} \leq 1. \tag{2.2}$$

Proof. Without loss of generality, assume that $a \leq b$.
Case $0 \leq a \leq \frac{1}{2}$. In order to prove above inequality, we show that $f(a) \leq \frac{1}{2}$, where $f(a) = a^{2b} = a^{2(1-a)}$. Since

$$f'(a) = a^{2(1-a)} g(a) > 0,$$

where $g(a) = \frac{2}{a} - 2 - 2lna$. (In fact, since $g'(a) = -\frac{2}{a^2} - \frac{2}{a} < 0$, $g(a)$ is strictly decreasing. Hence, we get $g(a) \geq g(1) = 0$ and $f'(a) > 0$.)

Therefore, $f(a)$ is strictly increasing as $a \in [0, \frac{1}{2}]$, and then $f(a) = a^{2b} \leq f(\frac{1}{2}) = \frac{1}{2}$. On the other hand, proof of $b^{2a} \leq \frac{1}{2}$ is similar to the previous case.

So, we have $a^{2b} + b^{2a} \leq \frac{1}{2} + \frac{1}{2} = 1$.
Case $\frac{1}{2} \leq a \leq 1$. Since $f(a) = a^{2(1-a)}$, we have

$$f'(a) = a^{2(1-a)}\frac{2}{a} - 2 - 2lna$$

and

$$f''(a) = a^{2(1-a)}[(\frac{2(1-a)}{a} - 2\ln a)^2 - \frac{2}{a^2} - \frac{2}{a}]$$

by a simple calculation.

Next, we show that $f''(a) \leq 0$. It suffices to show

$$(\frac{2(1-a)}{a} - 2\ln a)^2 \leq \frac{2}{a^2} + \frac{2}{a}$$

or

$$1 - a - a\ln a \leqslant \frac{\sqrt{2+2a}}{2}.$$

Owing to $h'(a) = -2 - lna$, where $h(a) = 1 - a - alna$, $h(a)$ is strictly decreasing as $a \in [\frac{1}{e^2}, 1]$, and then $h(a) \leqslant h(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2}\ln 2$.

On the other hand, $\frac{\sqrt{2+2a}}{2} \geqslant \frac{\sqrt{3}}{2} > \frac{1}{2} + \frac{1}{2}\ln 2$ is easy to prove in $a \in [\frac{1}{2}, 1]$. Therefore, we easily know that $f(a)$ is a concave function. Using Jessen's inequality, we have

$$f(a) + f(b) = a^{2b} + b^{2a} \leqslant 2f(\frac{a+b}{2}) = 2f(\frac{1}{2}) = 1.$$

Consider case $0 \leq a \leq \frac{1}{2}$ and case $\frac{1}{2} \leq a \leq 1$, we complete the proof.

In order to give a partial proof to Conjecture 4.3, we give following lemma.
Lemma 2.1[1]**.** If $0 < r \leq 2$, then

$$a^{ra} + b^{rb} \geq a^{rb} + b^{ra} \tag{2.3}$$

holds for all positive real numbers a and b.
Proposition 2.1. If a, b, c satisfy $\max\{a, b, c\} = a \geqslant 1$ or $\frac{1}{2} \leq c \leq b \leq a < 1$, then

$$a^{2a} + b^{2b} + c^{2c} \geq a^{2b} + b^{2c} + c^{2a}. \tag{2.4}$$

Proof. Case $\max\{a, b, c\} = a \geqslant 1$. By Lemma 2.1, we have

$$b^{2b} + c^{2c} \geqslant b^{2c} + c^{2b}.$$

Thus it suffice to prove

$$a^{2a} + c^{2b} \geqslant a^{2b} + c^{2a}.$$

For $a = b$, this inequality is an equality. Otherwise, for $a > b$, we substitute $x = c^{2b}, y = a^{2b}$ and $s = \frac{a}{b}$ (where $x \leq y, y \geq 1$, and $s \geq 1$) to rewrite the inequality as $f(x) \geq 0$ where

$$f(y) = y^s + x - y - x^s.$$

Since

$$f'(y) = sy^{s-1} - 1 \geq s - 1 \geq 0,$$

then $f(y)$ is strictly increasing for $x \leq y$, and therefore $f(y) \geq f(x) = 0$.
Case $\frac{1}{2} \leq c \leq b \leq a < 1$. By Lemma 2.1, we have

$$a^{2a} + b^{2b} \geqslant a^{2b} + b^{2a}.$$

Thus it suffice to prove

$$b^{2a} + c^{2c} \geqslant b^{2c} + c^{2a},$$

which is equivalent to $g(b) \geq g(c)$, where $g(x) = x^{2a} - x^{2c}$. The inequality is true if $g'(x) \geq 0$ for $c \leq x \leq b$. From

$$\begin{aligned} g'(x) &= 2ax^{2a-1} - 2cx^{2c-1} \\ &= 2c\ \ x^{2c-1}c^{2a-2c}(a - c^{1-2a+2c})' \end{aligned}$$

we need to show that $a - c^{1-2a+2c} \geq 0$. Since $0 \leq 1 - 2a + 2c \leq 1$, by Bernoulli's inequality, we have

$$\begin{aligned} &a - c^{1-2a+2c} \geqslant a - (c-1)(1 + (1 - 2a + 2c)) \\ &= (a - c)(2c - 1) \geqslant 0 \end{aligned}.$$

We finish the proof of Proposition 2.1.

In this way, we give a partial solution of Conjecture 4.3 in [1].

3. Open Problems

Problem 3.1. If a, b, c are positive real numbers, then

$$a^{a^a} + b^{b^b} + c^{c^c} \geqslant a^{b^c} + b^{c^a} + c^{a^b}. \tag{3.1}$$

Problem 3.2. If a, b, c are positive real numbers, then

$$a^{a^a} b^{b^b} c^{c^c} \geqslant a^{b^c} b^{c^a} c^{a^b}. \tag{3.2}$$

References

[1] V.Cîrtoaje, On some inequalities with power-exponential functions, *J. Inequal. Pure Appl. Math.***10**(1)(2009), Art. 21.

[2] L.Matejíčka, Solution of one conjecture on inequalities with power-exponential functions, *J. Inequal. Pure Appl. Math.***10**(3)(2009), Art. 72.