SOME GENERALIZED TRAPEZOIDAL VECTOR INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. On utilising the spectral representation of selfadjoint operators in Hilbert spaces, some generalized trapezoidal inequalities for continuous functions of such operators are given.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [9, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [9] and the references therein.

For other recent results see [5], [6], [7], [10], [11], [12] and [13].

Let U be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{R}$,

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it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d(\langle E_\lambda x,y\rangle), \tag{1.1}$$

for any $x,y\in H$. The function $g_{x,y}(\lambda):=\langle E_\lambda x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_x(\lambda):=\langle E_\lambda x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

With the notations introduced above, we have considered in the recent paper [8] the problem of bounding the error

$$\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle$$

in approximating $\langle f(A)x,y\rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle$, where x,y are vectors in the Hilbert space H and f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m,M]$.

We recall here only two such results. The first deals with the case of continuous functions of bounded variation and is incorporated in the following theorem [8]:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\begin{aligned}&\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right|\\&\le\frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.\\&\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\bigvee_m^M(f)\\&\le\frac{1}{2}\|x\|\,\|y\|\bigvee_m^M(f)\end{aligned}\tag{1.2}$$

for any $x,y\in H$.

The case of Lipschitzian functions is as follows [8]:

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then* *we have the inequality*

$$\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.3}$$
$$\le \frac{1}{2}L\int_{m-0}^{M}\Big[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$
$$+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\Big]d\lambda$$
$$\le\frac{1}{2}(M-m)L\|x\|\|y\|$$

for any $x,y\in H$.

In the present paper we are interested in providing error bounds for approximating $\langle f(A)x,y\rangle$ with the quantity

$$\frac{1}{M-m}\left[f(m)(M\langle x,y\rangle-\langle Ax,y\rangle)+f(M)(\langle Ax,y\rangle-m\langle x,y\rangle)\right] \tag{1.4}$$

where $x,y\in H$, which is a generalized trapezoid formula. Applications for some particular functions are provided as well.

2. Some Vector Inequalities

The following representation is of interest in itself and will be useful in deriving our inequalities later as well:

Lemma 1. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f:[m,M]\to\mathbb{C}$ *is a continuous function on* $[m,M]$*, then we have the representation*

$$\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle \tag{2.1}$$
$$=\int_{m-0}^{M}\langle E_tx,y\rangle\,df(t)-\frac{f(M)-f(m)}{M-m}\int_{m-0}^{M}\langle E_tx,y\rangle\,dt$$
$$=\int_{m-0}^{M}\left[\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle\,ds\right]df(t)$$

for any $x,y\in H$.

Proof. Integrating by parts and utilizing the spectral representation (1.1) we have

$$\begin{aligned}\int_{m-0}^{M}\langle E_tx,y\rangle\,df(t) &= f(M)\langle x,y\rangle-\int_{m-0}^{M}f(t)\,d\langle E_tx,y\rangle\\ &= f(M)\langle x,y\rangle-\langle f(A)x,y\rangle\end{aligned}$$

and

$$\int_{m-0}^{M}\langle E_tx,y\rangle\,dt=M\langle x,y\rangle-\langle Ax,y\rangle$$

for any $x,y\in H$.

Therefore

$$\begin{aligned}
&\int_{m-0}^{M} \langle E_t x, y\rangle \, df(t) - \frac{f(M) - f(m)}{M-m} \int_{m-0}^{M} \langle E_t x, y\rangle \, dt \\
&= f(M)\langle x, y\rangle - \langle f(A) x, y\rangle - \frac{f(M) - f(m)}{M-m} \left(M \langle x, y\rangle - \langle Ax, y\rangle \right) \\
&= \frac{1}{M-m} \left[f(m) \left(M \langle x, y\rangle - \langle Ax, y\rangle \right) + f(M) \left(\langle Ax, y\rangle - m \langle x, y\rangle \right) \right] \\
&- \langle f(A) x, y\rangle
\end{aligned}$$

for any $x, y \in H$, which proves the first equality in (2.1).

The second equality is obvious. □

The following result provides error bounds in approximating $\langle f(A) x, y\rangle$ by the generalized trapezoidal rule (1.4):

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

1. If $f : [m, M] \to \mathbb{C}$ is of bounded variation on $[m, M]$, then

$$\begin{aligned}
(2.2)\quad &\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y\rangle \right| \\
&\leq \sup_{t \in [m,M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) \right] \bigvee_{m}^{M} (f) \\
&\leq \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) \bigvee_{m}^{M} (f) \leq \|x\| \|y\| \bigvee_{m}^{M} (f)
\end{aligned}$$

for any $x, y \in H$.

2. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\begin{aligned}
(2.3)\quad &\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y\rangle \right| \\
&\leq L \int_{m}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) \right] dt \\
&\leq L(M-m) \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) \leq L(M-m) \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$.

3. If $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing on $[m, M]$, then

$$\begin{aligned}
(2.4)\quad &\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A) x, y\rangle \right| \\
&\leq \int_{m}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) \right] df(t) \\
&\leq \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)} x, y\right\rangle \right) [f(M) - f(m)] \leq \|x\| \|y\| [f(M) - f(m)]
\end{aligned}$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists, then the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \le \sup_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \tag{2.5}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Applying this property to the equality (2.1), we have

$$\begin{aligned} &\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A) x, y \rangle \right| \\ &\le \sup_{t \in [m,M]} \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle\, ds \right| \bigvee_m^M (f) \end{aligned} \tag{2.6}$$

for any $x, y \in H$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$\begin{aligned} &\langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle\, ds \\ &= \frac{1}{M - m} \left[\int_{m-0}^{t} (s - m)\, d\langle E_s x, y \rangle + \int_t^M (s - M)\, d\langle E_s x, y \rangle \right] \end{aligned} \tag{2.7}$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y \rangle$ is of bounded variation on $[m, M]$ for any $x, y \in H$, then on applying the inequality (2.5) once more, we get

$$\begin{aligned} &\left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^{M} \langle E_s x, y \rangle\, ds \right| \\ &\le \frac{1}{M - m} \left[\left| \int_{m-0}^{t} (s - m)\, d\langle E_s x, y \rangle \right| + \left| \int_t^M (s - M)\, d\langle E_s x, y \rangle \right| \right] \\ &\le \frac{t - m}{M - m} \bigvee_{m-0}^{t} \left(\langle E_{(\cdot)} x, y \rangle \right) + \frac{M - t}{M - m} \bigvee_t^M \left(\langle E_{(\cdot)} x, y \rangle \right) \end{aligned} \tag{2.8}$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Now, taking the supremum in (2.8) and taking into account that

$$\bigvee_{m-0}^{t} \left(\langle E_{(\cdot)} x, y \rangle \right), \bigvee_t^M \left(\langle E_{(\cdot)} x, y \rangle \right) \le \bigvee_{m-0}^{M} \left(\langle E_{(\cdot)} x, y \rangle \right)$$

for any $t \in [m, M]$ and for any $x, y \in H$, we deduce the first and the second inequality in (2.2).

If P is a nonnegative operator on H, i.e., $\langle Px, x \rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y \rangle|^2 \le \langle Px, x \rangle \langle Py, y \rangle \tag{2.9}$$

for any $x, y \in H$.

Further, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators that

$$\begin{aligned}
&\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&= \sup_d \left\{ \sum_{i=0}^{n-1} \left| \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y \right\rangle \right| \right\} \\
&\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle^{1/2} \right] \right\} := I.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
I &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}x, x\right\rangle\right) \right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}y, y\right\rangle\right) \right]^{1/2} = \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$. These prove the last part of (2.2).

Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \leq L \int_a^b |p(t)|\, dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.1) that

$$\begin{aligned}
&\left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y\rangle \right| \\
&\leq L \int_{m-0}^M \left| \langle E_t x, y\rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y\rangle\, ds \right| dt
\end{aligned} \tag{2.10}$$

for any $x, y \in H$.

Further on, by utilizing (2.7) we can state that

$$\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right|dt$$
$$\leq\frac{1}{M-m}\int_{m-0}^{M}\left[\left|\int_{m-0}^{t}(s-m)\,d\langle E_s x,y\rangle\right|+\left|\int_{t}^{M}(s-M)\,d\langle E_s x,y\rangle\right|\right]dt$$
$$\leq\int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\right]dt$$
$$\leq(M-m)\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)$$

for any $x,y\in H$, which proves the desired result (2.3).

From the theory of Riemann-Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b|p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right|\leq\int_a^b|p(t)|\,dv(t).$$

From the representation (2.1) we then have

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right|\tag{2.11}$$
$$\leq\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right|df(t)$$

for any $x,y\in H$.

Further on, by utilizing (2.7) we can state that

$$\int_{m-0}^{M}\left|\langle E_t x,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_s x,y\rangle\,ds\right|df(t)$$
$$\leq\frac{1}{M-m}\int_{m-0}^{M}\left[\left|\int_{m-0}^{t}(s-m)\,d\langle E_s x,y\rangle\right|+\left|\int_{t}^{M}(s-M)\,d\langle E_s x,y\rangle\right|\right]df(t)$$
$$\leq\int_{m-0}^{M}\left[\frac{t-m}{M-m}\bigvee_{m-0}^{t}\left(\langle E_{(\cdot)}x,y\rangle\right)+\frac{M-t}{M-m}\bigvee_{t}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\right]df(t)$$
$$\leq(f(M)-f(m))\bigvee_{m-0}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)$$

for any $x,y\in H$, which proves the desired result (2.4). □

A different approach for Lipschitzian functions is incorporated in:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral*

family. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then

$$\left|\left\langle\left[\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}\right]x,y\right\rangle-\langle f(A)x,y\rangle\right| \tag{2.12}$$
$$\leq L\|y\|\int_{m-0}^{M}\left\|E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|dt$$
$$\leq\frac{1}{2}L(M-m)\|x\|\|y\|$$

for any $x,y\in H$.

Proof. We will use the inequality (2.10) for which a different upper bound will be provided.

By the Schwarz inequality in H we have that

$$\int_{m-0}^{M}\left|\langle E_tx,y\rangle-\frac{1}{M-m}\int_{m-0}^{M}\langle E_sx,y\rangle ds\right|dt \tag{2.13}$$
$$=\int_{m-0}^{M}\left|\left\langle\left[E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right],y\right\rangle\right|dt$$
$$\leq\|y\|\int_{m-0}^{M}\left\|E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|dt$$

for any $x,y\in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$\int_{m-0}^{M}\left\|E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|dt \tag{2.14}$$
$$\leq(M-m)^{1/2}\left(\int_{m-0}^{M}\left\|E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|^2dt\right)^{1/2}$$

for any $x\in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$\frac{1}{M-m}\int_{m-0}^{M}\left\|E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|^2dt \tag{2.15}$$
$$=\frac{1}{M-m}\int_{m-0}^{M}\|E_tx\|^2dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|^2$$

and

$$\frac{1}{M-m}\int_{m-0}^{M}\|E_tx\|^2dt-\left\|\frac{1}{M-m}\int_{m-0}^{M}E_sxds\right\|^2 \tag{2.16}$$
$$=\frac{1}{M-m}\int_{m-0}^{M}\left\langle E_tx-\frac{1}{M-m}\int_{m-0}^{M}E_sxds,E_tx-\frac{1}{2}x\right\rangle dt$$

for any $x\in H$.

By (2.14), (2.15) and (2.16) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt \tag{2.17}$$
$$\leq (M-m)^{1/2}\left(\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt\right)^{1/2}$$

for any $x\in H$.

On making use of the Schwarz inequality in H we also have

$$\int_{m-0}^{M}\left\langle E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds, E_t x-\frac{1}{2}x\right\rangle dt \tag{2.18}$$
$$\leq \int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|\left\|E_t x-\frac{1}{2}x\right\|dt$$
$$=\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt,$$

where we used the fact that E_t are projectors, and in this case we have

$$\begin{aligned}\left\|E_t x-\frac{1}{2}x\right\|^2 &= \|E_t x\|^2-\langle E_t x,x\rangle+\frac{1}{4}\|x\|^2\\ &= \left\langle E_t^2 x,x\right\rangle-\langle E_t x,x\rangle+\frac{1}{4}\|x\|^2\\ &= \frac{1}{4}\|x\|^2\end{aligned}$$

for any $t\in[m,M]$ for any $x\in H$.

From (2.17) and (2.18) we get

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt \tag{2.19}$$
$$\leq (M-m)^{1/2}\left(\frac{1}{2}\|x\|\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt\right)^{1/2}$$

which is clearly equivalent with the following inequality of interest in itself

$$\int_{m-0}^{M}\left\|E_t x-\frac{1}{M-m}\int_{m-0}^{M}E_s x ds\right\|dt\leq\frac{1}{2}\|x\|(M-m) \tag{2.20}$$

for any $x\in H$.

This proves the last part of (2.12). □

3. Applications for Particular Functions

It is obvious that the above results can be applied for various particular functions. However, we will restrict here only to the power and logarithmic functions.

1. Consider now the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^p$ with $p\neq 0$. On applying Theorem 4 we can state the following proposition:

Proposition 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 \leq m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \tag{3.1}$$
$$\leq B_p \|y\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M - m} \int_{m-0}^{M} E_s x ds \right\| dt$$
$$\leq \frac{1}{2} B_p (M - m) \|x\| \|y\|$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ \\ m^{p-1} & \text{if } 0 < p < 1, m > 0 \end{cases}$$

and

$$B_p = (-p) m^{p-1} \text{ if } p < 0, m > 0.$$

2. The case of logarithmic function is as follows:

Proposition 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then for any $x, y \in H$ we have the inequalities*

$$\left| \left\langle \left[\frac{(M1_H - A) \ln m + (A - m1_H) \ln M}{M - m} \right] x, y \right\rangle - \langle \ln A x, y \rangle \right| \tag{3.2}$$
$$\leq \frac{1}{m} \|y\| \int_{m-0}^{M} \left\| E_t x - \frac{1}{M - m} \int_{m-0}^{M} E_s x ds \right\| dt$$
$$\leq \frac{1}{2} \left(\frac{M}{m} - 1 \right) \|x\| \|y\|.$$

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