ON THE SUPERADDITIVITY AND MONOTONICITY OF MAPPINGS ASSOCIATED WITH CAUCHY-SCHWARZ'S INEQUALITY IN 2-INNER PRODUCT SPACES I

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Abstract. Superadditivity and monotonicity of some mappings associated with the refinements of Cauchy-Schwarz's inequality in 2-inner product spaces are given.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book ([4]). Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 and $(\cdot,\cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x,x|z) \geq 0$,
$(x,x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x,x|z) = (z,z|x)$,
($2I_3$) $(x,y|z) = (y,x|z)$,
($2I_4$) $(\alpha x,y|z) = \alpha(x,y|z)$ for any real number α,
($2I_5$) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

$(\cdot,\cdot|\cdot)$ is called a *2-inner product* and $(X,(\cdot,\cdot|\cdot))$ is called a *2-inner product space*. Some basic properties of the 2-inner product $(\cdot,\cdot|\cdot)$ can be obtained as follows ([4]):

(1) For all $x,y,z \in X$,
$$|(x,y|z)| \leq \sqrt{(x,x|z)}\sqrt{(y,y|z)}.$$

(2) For all $x,y \in X$, $(x,y|y) = 0$ and $(x,y|0) = 0$.

(3) If $(X,(\cdot|\cdot))$ is an inner product space, then the 2-inner product $(\cdot,\cdot|\cdot)$ is defined on X by
$$(x,y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)$$

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for all $x, y, z \in X$.

Under the same assumptions over X, the real-valued function $\|\cdot,\cdot\|$ on $X \times X$ satisfying the following conditions:

($2N_1$) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
($2N_2$) $\|x, y\| = \|y, x\|$,
($2N_3$) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all real number α,
($2N_4$) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* ([7]). Some of the basic properties of the 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for every x, y in X and every real number α. Whenever a 2-inner product space $(X, (\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space on $(X, \|\cdot,\cdot\|)$ with the norm defined by $\|x, z\| = \sqrt{(x, x|z)}$ for all $x, z \in X$ and for any nonzero $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n)$ denote the subspace of X generated by $x_1, x_2, ..., x_n$.

Let $(X, (\cdot,\cdot|\cdot))$ be a 2-inner product space. If $(e_i)_{1\leq i\leq n}$ are linearly independent vectors in X and, for a given $z \in X$, $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, ..., n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1\leq i\leq n}$ is *z-orthonormal*), then the following inequality is the corresponding *Bessel's inequality* for the z-orthonormal family $(e_i)_{1\leq i\leq n}$ in X

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \leq \|x, z\|^2 \tag{1.1}$$

for any $x \in X$. For more details on this inequality, see [1], [3], [5]-[6], [8], [9].

For a 2-inner product space $(X, (\cdot,\cdot|\cdot))$, Cauchy-Schwarz's inequality

$$|(x, y|z)| \leq (x, x|z)^{1/2}(y, y|z)^{1/2} = \|x, z\|\|y, z\|, \tag{1.2}$$

a 2-dimensional analogue of Cauchy-Schwarz's inequality, holds ([4]). The following refinements of Cauchy-Schwarz's inequality in 2-inner product spaces has been obtains in [2]:

$$|(x, y|z)| \leq |(x, y|z) - (x, e|z)(e, y|z)| + |(x, e|z)(e, y|z)| \leq \|x, z\|\|y, z\|, \tag{1.3}$$

$$\begin{aligned} |(x, y|z) - (x, e|z)(e, y|z)| &\leq \Big((\|x, z\|^2 - |(x, e|z)|^2)(\|y, z\|^2 - |(y, e|z)|^2)\Big)^{1/2} \\ &\leq \Big(\|x, z\|\|y, z\| - |(x, e|z)(e, y|z)|\Big), \end{aligned} \tag{1.4}$$

$$\begin{aligned} |(x, y|z) - (x, e|z)(e, y|z)|^2 &\leq (\|x, z\|^2 - |(x, e|z)|^2)(\|y, z\|^2 - |(y, e|z)|^2) \\ &\leq \|x, z\|^2\|y, z\|^2 - |(x, y|z)|^2, \end{aligned} \tag{1.5}$$

$$\begin{aligned} |(x, y|z) - (x, e|z)(e, y|z)|^2 &\leq (\|x, z\|^2 - |(x, e|z)|^2)(\|y, z\|^2 - |(y, e|z)|^2) \\ &\leq \Big(\|x, z\|\|y, z\| - |(x, e|z)(e, y|z)|\Big)^2 \end{aligned} \tag{1.6}$$

for all $x, y, z, e \in X$ with $\|e, z\| = 1$ and $z \notin V(x, y, e)$.

In this paper, superadditivity and monotonicity of some mappings associated with the refinements of Cauchy-Schwarz's inequality in 2-inner product spaces are given.

2. Refinements of Cauchy-Schwarz's inequality

In this section, we shall establish some results on the refinements of Cauchy-Schwarz's inequality in 2-inner product spaces.

Lemma 2.1([2])**.** Let X be a linear 2-normed spaces and $x, y, z, u, v \in X$ with $z \notin V(x, y, u, v)$ be such that

$$\|u, z\|^2 \le 2(x, u|z) \quad and \quad \|v, z\|^2 \le 2(y, v|z).$$

Then we have the inequality:

$$\begin{aligned} &\Big(2(x, u|z) - \|u, z\|^2\Big)^{1/2}\Big(2(y, v|z) - \|v, z\|^2\Big)^{1/2} \\ &\quad + \Big|(x, y|z) - (x, v|z) - (u, y|z) + (u, v|z)\Big| \\ &\le \|x, z\|\|y, z\|. \end{aligned} \tag{2.1}$$

Theorem 2.2. Let X be a linear 2-normed space, $x, y, z \in X$ with $z \notin V(x, y)$ and $(e_i)_{1 \le i \le n}$ be a family of z-orthonomal vectors in X. Then we have the following inequality:

$$\begin{aligned} |(x, y|z)| &\le \left|(x, y|z) - \sum_{i=1}^n (x, e_i|z)(e_i, y|z)| + \sum_{i=1}^n |(x, e_i|z)(e_i, y|z)\right| \\ &\le \|x, z\|\|y, z\|. \end{aligned} \tag{2.2}$$

Proof. Let $u = \sum_{i=1}^n (x, e_i|z)e_i$ and $v = \sum_{i=1}^n (y, e_i|z)e_i$. Then we have

$$2(x, u|z) - \|u, z\|^2 = \sum_{i=1}^n |(x, e_i|z)|^2 \ge 0 \quad \text{and} \quad 2(y, v|z) - \|v, z\|^2 = \sum_{i=1}^n |(y, e_i|z)|^2 \ge 0.$$

We also have

$$\Big|(x, y|z) - (x, v|z) - (u, y|z) + (u, v|z)\Big| = \left|(x, y|z) - \sum_{i=1}^n (x, e_i|z)(e_i, y|z)\right|.$$

Thus, by the inequality (2.1) and triangle inequality, we have the desired inequality (2.2). This completes the proof. □

Theorem 2.3. With the assumptions of Theorem 2.2, we have the following inequality:

$$\begin{aligned} 0 &\le \left|(x, y|z) - \sum_{i=1}^n (x, e_i|z)(e_i, y|z)\right|^2 \\ &\le \left(\|x, z\|^2 - \sum_{i=1}^n |(x, e_i|z)|^2\right)\left(\|y, z\|^2 - \sum_{i=1}^n |(y, e_i|z)|^2\right) \\ &\le \left(\|x, z\|\|y, z\| - \sum_{i=1}^n |(x, e_i|z)(e_i, y|z)|\right)^2 \end{aligned} \tag{2.3}$$

Proof. By Cauchy-Schwarz's inequality (1.2) in a 2-inner product space X, we have

$$\left|\left(x-\sum_{i=1}^{n}(x,e_i|z)e_i,\, y-\sum_{i=1}^{n}(y,e_i|z)e_i \,\Big|\, z\right)\right|^2 \le \left\|x-\sum_{i=1}^{n}(x,e_i|z)e_i\, z\right\|^2\left\|y-\sum_{i=1}^{n}(y,e_i|z)e_i, z\right\|^2,$$

which is equivalent with

$$\left|(x,y|z)-\sum_{i=1}^{n}(x,e_i|z)(y,e_i|z)\right|^2 \le \left(\|x,z\|^2-\sum_{i=1}^{n}|(x,e_i|z)|^2\right)\left(\|y,z\|^2-\sum_{i=1}^{n}(y,e_i|z)\|^2\right)$$

from where we have the first part of inequality (2.3).

In order to prove the second part of inequality (2.3), using Aczél's inequality ([10]) we get

$$\begin{aligned}(\|x,z\|^2-\sum_{i=1}^{n}|(x,e_i|z)|^2)(\|y,z\|^2-\sum_{i=1}^{n}|(y,e_i|z)|^2)\\ \le\left(\|x,z\|\|y,z\|-\sum_{i=1}^{n}|(x,e_i|z)||(y,e_i|z)|\right)^2,\end{aligned}$$

and so the second part of inequality (2.3) holds. This completes the proof. □

Corollary 2.4. With the assumptions of Theorem 2.2, we have the following inequality:

$$\begin{aligned}&\left(\|x+y,z\|^2-|\sum_{i=1}^{n}(x+y,e_i|z)|^2\right)^{1/2}\\ &\le\left(\|x,z\|^2-\sum_{i=1}^{n}|(x,e_i|z)|^2\right)^{1/2}+\left(\|y,z\|^2-\sum_{i=1}^{2}|(e_i,y|z)^2\right)^{1/2}\end{aligned} \tag{2.4}$$

Theorem 2.5. With the assumptions of Theorem 2.2, we have the following inequality:

$$\begin{aligned}&\|x,z\|\|y,z\|-|(x,y|z)|\\ &\ge\left(\sum_{i=1}^{n}|(x,e_i|z)|^2\right)^{1/2}\left(\sum_{i=1}^{n}|(e_i,y|z)|^2\right)^{1/2}-\left|\sum_{i=1}^{n}(x,e_i|z)(e_i,y|z)\right|\ge 0.\end{aligned} \tag{2.5}$$

Proof. By the first part of inequality (2.3),

$$\begin{aligned}&\left|(x,y|z)-\sum_{i=1}^{n}(x,e_i|z)(e_i,y|z)\right|^2\\ &\le\left(\|x,z\|^2-\sum_{i=1}^{n}|(x,e_i|z)|^2\right)\left(\|y,z\|^2-\sum_{i=1}^{n}|(y,e_i|z)|^2\right)\end{aligned} \tag{2.6}$$

and using the elementary inequality

$$(a^2-c^2)(b^2-d^2)\le(ac-bd)^2$$

for all $a, b, c, d \in R$, we have

$$\left(\|x, z\|^2 - \sum_{i=1}^{n} |(x, e_i|z)|^2\right)\left(\|y, z\|^2 - \sum_{i=1}^{n} |(y, e_i|z)|^2\right) \leq \left[\|x, z\|\|y, z\| - \left(\sum_{i=1}^{n} |(x, e_i|z)|^2\right)^{1/2}\left(\sum_{i=1}^{n} |(y, e_i|z)|^2\right)^{1/2}\right]^2. \tag{2.7}$$

By Bessel's inequality (1.1), we also have

$$\|x, z\|\|y, z\| \geq \left(\sum_{i=1}^{n} |(x, e_i|z)|^2\right)^{1/2}\left(\sum_{i=1}^{n} |(y, e_i|z)|^2\right)^{1/2}$$

and by the inequalities (2.6) and (2.7), we yield the following inequality:

$$\left|(x, y|z) - \sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)\right| \leq \|x, z\|\|y, z\| - \left(\sum_{i=1}^{n} |(x, e_i|z)|^2\right)^{1/2}\left(\sum_{i=1}^{n} |(y, e_i|z)|^2\right)^{1/2}. \tag{2.8}$$

Since

$$|(x, y|z)| - \left|\sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)\right| \leq \left|(x, y|z) - \sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)\right|,$$

we obtain the desired inequality (2.5). This completes the proof. □

Corollary 2.6. With the assumptions of Theorem 2.2, we have the following inequality:

$$(\|x, z\| + \|y, z\|)^2 - \|x + y, z\|^2 \geq \left[\left(\sum_{i=1}^{n} |(x, e_i|z)|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |(e_i, y|z)|^2\right)^{1/2}\right]^2 - \sum_{i=1}^{n} |(x, e_i|z) + (e_i, y|z)|^2 \geq 0. \tag{2.9}$$

3. Superadditivity and monotonicity of some mappings

In this section, we shall derive striking superadditivity and monotonicity properties of mappings associated with the refinements (2.3) and (2.5) of Cauchy-Schwarz's inequality in 2-inner product spaces.

Let X be a 2-inner product space and $P(N)$ denote the class of all finite indices of N. Fixed a family $(e_i)_{i \in N}$ of a z-orthonormal vectors in X. We can consider the index set mappings $\alpha, \beta : P(N) \times X^3 \to R$ defined by

$$\alpha(I, x, y, z) = \left(\sum_{i \in I} |(x, e_i|z)|^2 \sum_{i \in I} |(y, e_i|z)|^2\right)^{1/2} - \left|\sum_{i \in I}(x, e_i|z)(e_i, y|z)\right|$$

and

$$\beta(I,x,y,z) = \left[\left(\|x,z\|^2 - \sum_{i\in I}|(x,e_i|z)|^2\right)\left(\|y,z\|^2 - \sum_{i\in I}|(y,e_i|z)|^2\right)\right]^{1/2} - \left|(x,y|z) - \sum_{i\in I}(x,e_i|z)(e_i,y|z)\right|.$$

for all $x,y,z \in X$ with $z \notin V(x,y)$ and $I \in P(N)$.

Theorem 3.1. Let X be a 2-inner product space and $(e_i)_{i\in N}$ be a family of z-orthonormal vectors in X. Then we have

$$\|x,z\|\|y,z\| - |(x,y|z)| \geq \alpha(I,x,y,z) + \beta(I,x,y,z) \geq 0 \tag{3.1}$$

for all $x,y,z \in X$ with $z \notin V(x,y)$ and $I \in P(N)$.

Proof. Using elementary inequalities

$$0 \leq ab + cd \leq (a^2+c^2)^{1/2}(b^2+d^2)^{1/2} \qquad \text{for} \quad a,b,c,d \geq 0$$

and by triangle inequality, we have

$$\begin{aligned}
&\alpha(I,x,y,z) + \beta(I,x,y,z) \\
&= (\sum_{i\in I}|(x,e_i|z)|^2 \sum_{i\in I}|(y,e_i|z)|^2)^{1/2} \\
&\quad + \left[\left(\|x,z\|^2 - \sum_{i\in I}|(x,e_i|z)|^2\right)\left(\|y,z\|^2 - \sum_{i\in I}|(y,e_i|z)|^2\right)\right]^{1/2} \\
&\quad - \left|\sum_{i\in I}|(x,e_i|z)(y,e_i|z)\right| - \left|(x,y|z) - \sum_{i\in I}(x,e_i|z)(e_i,y|z)\right| \\
&\leq \left\{\left[(\sum_{i\in I}|x,e_i|z)|^2)^{1/2}\right]^2 + \left[(\|x,z\|^2 - \sum_{i\in I}|(x,e_i|z)|^2)^{1/2}\right]^2\right\} \\
&\quad \times \left\{\left[(\sum_{i\in I}|(y,e_i|z)|^2)^{1/2}\right]^2 + \left[(\|y,z\|^2 - \sum_{i\in I}|(y,e_i|z)|^2)^{1/2}\right]^2\right\} \\
&\quad - \left|\sum_{i\in I}(x,e_i|z)(e_i,y|z) + (x,y|z) - \sum_{i\in I}(x,e_i|z)(e_i,y|z)\right| \\
&= \|x,z\|\|y,z\| - |(x,y|z)|
\end{aligned}$$

for all $x,y,z \in X$ with $z \notin V(x,y)$ and $I \in P(N)$. Thus, the inequality (3,1) is holds. This completes the proof. □

Theorem 3.2. With the assumptions of Theorem 3.1, we have
(i) For all $I,J \in P(N)$ with $I \cap J = \phi$

$$\alpha(I\cup J,x,y,z) \geq \alpha(I,x,y,z) + \alpha(J,x,y,z) \geq 0 \tag{3.2}$$

for all $x,y,z \in X$ with $z \notin V(x,y)$, i.e., the mapping $\alpha(\cdot,x,y,z)$ is superadditivie on $P(N)$.

(ii) For all $I, J \in P(N)$ with $I \supset J (\neq \phi)$

$$\alpha(I, x, y, z) \geq \alpha(J, x, y, z) \geq 0 \tag{3.3}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, i.e., the mapping $\alpha(\cdot, x, y, z)$ is monotonic nondecreasing on $P(N)$.

Proof. (i) Suppose that $I, J \in P(N)$ with $I \cap J = \phi$. Then we have

$$\begin{aligned}
&\alpha(I \cup J, x, y, z) \\
&= (\sum_{k \in I \cup J} |(x, e_k|z)|^2 \sum_{k \in I \cup J} |(y, e_k|z)|^2)^{1/2} - |\sum_{k \in I \cup J} (x, e_k|z)(e_k, y|z)| \\
&= (\sum_{i \in I} |(x, e_i|z)|^2 + \sum_{j \in J} |(x, e_j|z)|^2)^{1/2} (\sum_{i \in I} |(y, e_i|z)|^2 \sum_{j \in J} |(y, e_j|z)|^2)^{1/2} \\
&\qquad - |\sum_{i \in I} (x, e_i|z)(e_i, y|z) + \sum_{j \in J} (x, e_j|z)(e_j, y|z)| \\
&\geq (\sum_{i \in I} |(x, e_i|z)|^2)^{1/2} (\sum_{i \in I} |(y, e_i|z)|^2)^{1/2} + (\sum_{j \in J} |(x, e_j|z)|^2)^{1/2} (\sum_{j \in J} |(y, e_j|z)|^2)^{1/2} \\
&\qquad - |\sum_{i \in I} (x, e_i|z)(e_i, y|z)| + |\sum_{j \in J} (x, e_j|z)(e_j, y|z)| \\
&= \alpha(I, x, y, z) + \alpha(J, x, y, z)
\end{aligned}$$

for all $x, y, z \in X$ and $z \notin V(x, y)$, and so the inequality (3.2) is proved.

(ii) Suppose that $I, J \in P(N)$ with $I \supset J (\neq \phi)$ and $I \neq J$. Then, by (i) we have

$$\alpha(I, x, y, z) = \alpha((I \setminus J) \cup J, x, y, z) \geq \alpha(I \setminus J, x, y, z) + \alpha(J, x, y, z),$$

which gives

$$\alpha(I, x, y, z) - \alpha(J, x, y, z) \geq \alpha(I \setminus J, x, y, z) \geq 0$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, and so the inequality (3.3) is proved. This completes the proof. □

Corollary 3.3. Let $\{e_i\}_{i \in N}$ be a z-orthonormal sequence of vectors in a 2-inner product space X. Put

$$\alpha_n(x, y, z) = \alpha(I_n, x, y, z)$$

where $I_n = \{1, 2, \dots, n\} \in P(N)$. Then we have:

(i) For $n \geq 2$ and $x, y, z \in X$ with $z \notin V(x, y)$

$$\begin{aligned}
\alpha_n(x, y, z) \geq \max_{1 \leq i < j \leq n} \Bigg[&\left(|(x, e_i|z)|^2 + |(x, e_j|z)|^2 \right)^{1/2} \left(|(y, e_i|z)|^2 + |(y, e_j|z)|^2 \right)^{1/2} \\
&- |(x, e_i|z)(e_i, y|z) + (x, e_j|z)(e_j, y|z)| \Bigg] \geq 0.
\end{aligned}$$

(ii) For $n \geq 2$ and $x, y, z \in X$ with $z \notin V(x, y)$

$$\begin{aligned}
0 &= \alpha_1(x, y, z) \leq \alpha_2(x, y, z) \leq \cdots \leq \alpha_n(x, y, z) \leq \alpha_{n+1}(x, y, z) \leq \cdots \\
&\leq \left(\sum_n |(x, e_n|z)|^2 \sum_n |(y, e_n|z)|^2 \right)^{1/2} - \left| \sum_n (x, e_n|z)(y, e_n|z) \right| \\
&= \sup_{n \geq 1} \alpha_n(x, y, z) = \lim_{n \to \infty} \alpha_n(x, y, z) \\
&\leq \|x, z\| \|y, z\| - |(x, y|z)|,
\end{aligned}$$

which gives another type of refinement of Cauchy-Schwarz's inequality.

Theorem 3.4. With the assumptions of Theorem 3.1,
(i) For all $I, J \in P(N)$ with $I \cap J = \phi$ we have the inequality

$$0 \le \beta(I \cup J, x, y, z) + \frac{1}{2}[\alpha(I, x, y, z) + \alpha(J, x, y, z)] \le \frac{1}{2}[\beta(I, x, y, z) + \beta(J, x, y, z)] \tag{3.4}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$.
(ii) For all $I, J \in P(N)$ with $I \supset J (\neq \phi)$ we have the inequality

$$0 \le \beta(I, x, y, z) \le \beta(J, x, y, z) \tag{3.5}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, i.e., the mapping $\beta(\cdot, x, y, z)$ is monotonic noninecreasing on $P(N)$.

Proof. Let $I, J \in P(N)$ with $I \cap J = \phi$. Using a similar argument as in Theorem 3.1, we have

$$\begin{aligned}
&\beta(I \cup J, x, y, z) \\
&= \left(\|x, z\|^2 - \sum_{i \in I} |(x, e_i|z)|^2 - \sum_{j \in J} |(x, e_j|z)|^2\right)^{1/2} \\
&\qquad \times \left(\|y, z\|^2 - \sum_{i \in I} |(y, e_i|z)|^2 - \sum_{j \in J} |(y, e_j|z)|^2\right)^{1/2} \\
&\qquad - \left|(x, y|z) - \sum_{i \in I} (x, e_i|z)(e_i, y|z) - \sum_{j \in J} (x, e_j|z)(e_j, y|z)\right| \\
&\le \left[\left(\|x, z\|^2 - \sum_{i \in I} |(x, e_i|z)|^2\right)\left(\|y, z\|^2 - \sum_{i \in I} |(y, e_i|z)|^2\right)\right]^{1/2} \\
&\qquad - \left(\sum_{j \in J} |(x, e_j|z)|^2\right)^{1/2}\left(\sum_{j \in J} |(y, e_j|z)|^2\right)^{1/2} \\
&\qquad - \left|(x, y|z) - \sum_{i \in I} |(x, e_i|z)(e_i, y|z)\right| + \left|\sum_{j \in J} |(x, e_j|z)(e_j, y|z)\right| \\
&\le \beta(I, x, y, z) - \alpha(J, x, y, z)
\end{aligned} \tag{3.6}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. By interchanging I and J in the inequality (3.6),

$$\beta(J \cup I, x, y, z) \le \beta(J, x, y, z) - \alpha(I, x, y, z). \tag{3.7}$$

Thus, by addition these inequalities (3.6) and (3.7) we can deduce the inequality (3.4).
(ii) Suppose that $I \supset J (\neq \phi)$ and $J \neq I$. Then, by the inequality (3.6) we have

$$\beta((I \setminus J) \cup J, x, y, z) + \alpha(I \setminus J, x, y, z) \le \beta(J, x, y, z),$$

which gives

$$\beta(J, x, y, z) - \beta(I, x, y, z) \ge \alpha(I \setminus J, x, y, z) \ge 0$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. Thus, the inequality (3.5) holds. This completes the proof. □

Remark. From the inequality (3.6), we have

$$\alpha(I, x, y, z) \leq \beta(J, x, y, z)$$

for all $I, J \in P(N)$ with $I \cap J = \phi$ and $x, y, z \in X$ with $z \notin V(x, y)$.

Corollary 3.5. Let $\{e_i\}_{i\in N}$ be a z-orthonormal sequence of vectors in a 2-inner product space X. Put

$$\beta_n(x, y, z) = \beta(I_n, x, y, z)$$

where $I_n = \{1, 2, \dots, n\} \in P(N)$. Then we have

(i) For $n \geq 2$ and $x, y, z \in X$ with $z \notin V(x, y)$

$$\begin{aligned}\beta_n(x, y, z) \leq \min_{1\leq i<j\leq n} \Bigg[& \Big(\|x, z\|^2 - |(x, e_i|z)|^2 - |(x, e_j|z)|^2 \Big)^{1/2} \\ & \times \Big(\|y, z\|^2 - |(y, e_i|z)|^2 - |(y, e_j|z)|^2 \Big)^{1/2} \\ & - \Big| (x, y|z) - (x, e_i|z)(e_i, y|z) - (x, e_j|z)(e_j, y|z) \Big| \Bigg].\end{aligned}$$

(ii) For $n \geq 2$ and $x, y, z \in X$ with $z \notin V(x, y)$

$$\begin{aligned}&\|x, z\|\|y, z\| - |(x, y|z)| \geq \beta_1(x, y, z) \geq \cdots \geq \beta_n(x, y, z) \geq \beta_{n+1}(x, y, z) \geq \cdots \\ &\geq \inf_{n\geq 1} \beta_n(x, y, z) = \lim_{n\to\infty} \beta_n(x, y, z) \\ &= \Big(\|x, z\|^2 - \sum_n |(x, e_i|z)|^2 \Big)^{1/2} \Big(\|y, z\|^2 - \sum_n |(y, e_i|z)|^2 \Big)^{1/2} \\ &\quad - |(x, y|z) - \sum_n (x, e_i|z)(e_j, y|z)|,\end{aligned}$$

which gives another type of refinement of Cauchy-Schwarz's inequality.

Theorem 3.6. With the assumptions of Theorem 3.1, we have the following inequality:

$$\begin{aligned}&\|x, z\|\|y, z\| + \beta(I \cup J, x, y, z) \\ &\leq \frac{1}{2}\Big[\beta(I, x, y, z) + \beta(J, x, y, z)\Big] + \frac{1}{2}\Big[\alpha(I, x, y, z) + \alpha(J, x, y, z)\Big]\end{aligned} \tag{3.8}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$ and $I, J \in P(N)$ with $I \cap J = \phi$.

Proof. Let $I, J \in P(N)$ with $I \cap J = \phi$. From the inequality (3.6) and the following inequality

$$(a - b)^{\frac{1}{2}} \geq a^{\frac{1}{2}} - b^{\frac{1}{2}} \geq 0 \quad \text{for} \quad a \geq b \geq 0,$$

we have

$$
\begin{aligned}
&\beta(I \cup J, x, y, z) \\
&= \left(\|x,z\|^2 - \sum_{i\in I} |(x, e_i|z)|^2 - \sum_{j\in J} |(x, e_j|z)|^2\right)^{1/2} \\
&\qquad \times \left(\|y,z\|^2 - \sum_{i\in I} |(y, e_i|z)|^2 - \sum_{j\in J} |(y, e_j|z)|^2\right)^{1/2} \\
&\quad - \left|(x,y|z) - \sum_{i\in I}(x, e_i|z)(e_i, y|z) - \sum_{j\in J}(x, e_j|z)(e_j, y|z)\right| \\
&\geq \left[\left(\|x,z\|^2 - \sum_{i\in I} |(x, e_i|z)|^2\right)^{1/2} - \left(\sum_{j\in J} |(x, e_j|z)|^2\right)^{1/2}\right] \\
&\qquad \times \left[\left(\|y,z\|^2 - \sum_{i\in I} |(y, e_i|z)|^2\right)^{1/2} - \left(\sum_{j\in J} |(y, e_j|z)|^2\right)^{1/2}\right] \\
&\quad - \left|(x,y|z) - \sum_{i\in I} |(x, e_i|z)(e_i, y|z)\right| + \left|\sum_{j\in J} |(x, e_j|z)(e_j, y|z)\right| \\
&\geq \beta(I, x, y, z) + \alpha(J, x, y, z) - \mu(I, J, x, y, z)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu(I, J, x, y, z) &= \left(\|y,z\|^2 - \sum_{i\in I} |(y, e_i|z)|^2\right)^{1/2} \left(\sum_{j\in J} |(x, e_j|z)|^2\right)^{1/2} \\
&\qquad + \left(\|x,z\|^2 - \sum_{i\in I} |(x, e_i|z)|^2\right)^{1/2} \left(\sum_{j\in J} |(y, e_j|z)|^2\right)^{1/2}.
\end{aligned}
$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. Hence, we have

$$
\beta(I \cup J, x, y, z) + \mu(I, J, x, y, z) \geq \beta(I, x, y, z) + \alpha(J, x, y, z). \tag{3.9}
$$

By interchanging I and J in the inequality (3.9), we have

$$
\beta(J \cup I, x, y, z) + \mu(J, I, x, y, z) \geq \beta(J, x, y, z) + \alpha(I, x, y, z). \tag{3.10}
$$

Adding these inequalities (3.9) and (3.10), we have

$$
\begin{aligned}
&2\beta(I \cup J, x, y, z) + \mu(I, J, x, y, z) + \mu(J, I, x, y.z) \\
&\quad \geq \beta(I, x, y, z) + \beta(J, x, y, z) + \alpha(I, x, y, z) + \alpha(J, x, y, z).
\end{aligned} \tag{3.11}
$$

Now, from Cauchy-Schwarz's inequality in a 2-inner product space X we have

$$
\mu(I, J, x, y, z) + \mu(J, I, x, y, z) \leq (2\|y,z\|^2)^{1/2}(2\|x,z\|^2)^{1/2} = 2\|x,z\|\|y,z\|. \tag{3.12}
$$

Thus, the inequalities (3.11) and (3.12) reduces to the desired inequality (3.8). This completes the proof. □

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