ON ACZÉL'S TYPE INEQUALITY FOR GRAMIANS IN 2-INNER PRODUCT SPACES

SEONG SIK KIM*, YEOL JE CHO† AND SEVER S. DRAGOMIR

ABSTRACT. In this paper, some results related to Aczél's type inequality for Gramians in terms of Kurepa's and Hadamard's inequality in 2-inner product spaces are given.

I. Introduction

In 1956, J. Aczél proved the following inequality, known in literature as Aczél's inequality ([10]):

Theorem A. Let $a = (a_1, a_2, \dots, a_m)$ and $b = (b_1, b_2, \dots, b_m)$ be two sequences of real numbers such that

$$a_1^2 - a_2^2 - \cdots - a_m^2 > 0 \quad \text{or} \quad b_1^2 - b_2^2 - \cdots - b_m^2 > 0.$$

Then we have the inequality

$$\begin{aligned}(a_1^2 - a_2^2 - \cdots - a_m^2)(b_1^2 - b_2^2 - \cdots - b_m^2) \\ \leq (a_1 b_1 - a_2 b_2 - \cdots - a_m b_m)^2\end{aligned} \tag{1.1}$$

with the equality if and only if the sequences a and b are proportional.

In [9], S. Kurepa proved the following inequality of Aczél type which holds in Hilbert spaces :

Theorem B. Let X be a real Hilbert space and c a unit vector in X. Suppose that $a, b \in X$ are given vectors such that

$$(u^2 - \|a_o\|^2) \times (v^2 - \|b_o\|^2) \geq 0$$

† : Corresponding author : Y.J. Cho

2000 Mathematics Subject Classification : Primary 46B05, 46C05; Secondary 26D15, 26D10

Key words and Phrases : Aczél inequality, Kurepa's inequality, Hadamard's inequality, Gramians

*This work was supposed from from Dongeui University, 2009

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

where $u = (a, c), v = (b, c)$, $a_o = a - uc$ and $b_o = b - vc$. Then

$$\left(u^2 - \|a_o\|^2\right) \times \left(v^2 - \|b_o\|^2\right) \geq (uv - (a_o, b_o))^2. \tag{1.2}$$

If a and b are independent and strict inequality holds in (1.2), then strict inequality also holds in (1.2).

S.S. Dragomir ([6]) proved the generalization results of Aczél's inequality:

Theorem C. Let $(X, (\cdot, \cdot))$ be an inner product space over the real and complex numbers field K and $a, b, c \in R$ satisfy the following condition:

$$a, c > 0 \quad \text{and} \quad b^2 \geq ac.$$

Then, for all $x, y \in X$ with $a \geq \|x\|^2$ or $c \geq \|y\|^2$, we have the inequality

$$\begin{aligned} &(a - \|x\|^2)(c - \|y\|^2) \\ &\leq \min\Big\{(b \pm Re(x, y))^2, (b \pm |Re(x, y)|)^2, \\ &\qquad (b \pm Im(x, y))^2, (b \pm |Im(x, y)|)^2, (b \pm |(x, y)|)^2\Big\}. \end{aligned} \tag{1.3}$$

Let X be a linear space of dimension greater than 1. and $(\cdot, \cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x, x|z) \geq 0$,
$(x, x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x, x|z) = (z, z|x)$,
($2I_3$) $(x, y|z) = (y, x|z)$,
($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any real number α,
($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(.,.|.)$ is called a *2-inner product* and $(X, (\cdot, \cdot|\cdot))$ a *2-inner product space* ([2]).

Some basic properties of the 2-inner product $(\cdot, \cdot|\cdot)$ are as follows ([2]):

(1) For all $x, y, z \in X$,

$$|(x, y|z)| \leq \sqrt{(x, x|z)}\sqrt{(y, y|z)}.$$

(2) For all $x, y \in X$, $(x, y|y) = 0$.

(3) If $(X, (\cdot|\cdot))$ is an inner product space, then the 2-inner product $(\cdot, \cdot|\cdot)$ is defined on X by

$$(x, y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)$$

for all $x, y, z \in X$.

Under the same assumptions over X, the real-valued function $\|\cdot,\cdot\|$ on $X \times X$ satisfying the following conditions:

($2N_1$) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
($2N_2$) $\|x, y\| = \|y, x\|$,
($2N_3$) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all real number α,
($2N_4$) $\|x, y + z\| \le \|x, y\| + \|x, z\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ a *linear 2-normed space* ([8]). Some of the basic properties of the 2-norms are that they are non-negative and $\|x, y+\alpha x\| = \|x, y\|$ for every x, y in X and every real number α.

For any nonzero $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n)$ denote the subspace of X generated by $x_1, x_2, ..., x_n$. Whenever the notation $V(x_1, x_2, ..., x_n)$ is used, by it will understood $x_1, x_2, ..., x_n$ to be linearly independent.

Let $(X, (\cdot,\cdot|\cdot))$ be a 2-inner product space. For any elements $x_1, \dots, x_m, z \in X$ we define the Gram's matrix $G_z(x_1, \dots, x_m)$ of $x_1, ..., x_m$ with respect to z by

$$G_z(x_1, \dots, x_m) = \begin{pmatrix} (x_1, x_1|z) & \dots & (x_1, x_m|z) \\ \vdots & \ddots & \vdots \\ (x_m, x_1|z) & \dots & (x_m, x_m|z) \end{pmatrix}$$

and Gram's determinant $\Gamma_z(x_1, \dots, x_m)$ of $x_1, ..., x_m$ with respect to z by

$$\begin{aligned} \Gamma_z(x_1, \dots, x_m) &= \det G_z(x_1, \dots, x_m) \\ &= \det \begin{pmatrix} (x_1, x_1|z) & \dots & (x_1, x_m|z) \\ \vdots & \ddots & \vdots \\ (x_m, x_1|z) & \dots & (x_m, x_m|z) \end{pmatrix}. \end{aligned}$$

Then we have the following inequality:

$$\Gamma_z(x_1, \dots, x_m) \ge 0. \tag{1.4}$$

The equality holds in (1.4) if and only if $x_1, ..., x_n, z$ are linearly dependent. The inequality (1.4) is said to be *Gram's inequality* in 2-inner product spaces.

In the inequality (1.4), if $n = 2$, then

$$(x_1, x_1|z)(x_2, x_2|z) - |(x_1, x_2|z)|^2 \ge 0$$

with the equality if and only if x_1, x_2, z are linearly dependent. This inequality can be regards as a *generalization of Cauchy-Buniakowski-Schwarz's inequality* in 2-inner product spaces.

Some inequalities which involve Gram's determinants are given ([5]):

$$\frac{\Gamma_z(x_1, \dots, x_m)}{\Gamma_z(x_1, \dots, x_k)} \le \frac{\Gamma_z(x_2, \dots, x_m)}{\Gamma_z(x_2, \dots, x_k)} \le \dots \le \Gamma_z(x_{k+1}, \dots, x_m), \tag{1.5}$$

$$\Gamma_z(x_1,\ldots,x_k,x_{k+1}\ldots,x_m)\le\Gamma_z(x_1,\ldots,x_k)\Gamma_z(x_{k+1},\ldots,x_m), \tag{1.6}$$

$$\begin{aligned}&\Gamma_z(x_1+y_1,x_2,\ldots,x_m)^{1/2}\\&\qquad\le\Gamma_z(x_1,x_2,\ldots,x_m)^{1/2}+\Gamma_z(y_1,x_2,\ldots,x_m)^{1/2}.\end{aligned} \tag{1.7}$$

For further details on some properties of Aczél's and Gram's inequalities of 2-inner product spaces are given by [1], [3] - [5], [7] - [9].

In this paper, some results related to Aczél's type inequality for Gramians in terms of Kurepa's and Hadamard's inequality, which generalize and extend the corresponding results of S.S. Dragomir [6] to 2-inner product spaces are given.

II. Aczél's type inequality and Kurepa's inequality

Let X be a 2-inner product space and $x_1,\ldots,x_m,y_1,\ldots y_m,z\in X$. We can define the determinant $\tilde\Gamma_z(x_1,y_1,...,x_m,y_m)$ by

$$\tilde\Gamma_z(x_1,y_1,...,x_m,y_m)=\det\begin{pmatrix}(x_1,y_1|z)&\ldots&(x_1,y_m|z)\\\vdots&\ddots&\vdots\\(x_m,y_1|z)&\ldots&(x_m,y_m|z)\end{pmatrix}$$

for $z\notin V(x_1,...,x_m,y_1,...,y_m)$. Note that if $y_1=x_1,...,y_m=x_m$, then

$$\tilde\Gamma_z(x_1,y_1,...,x_m,y_m)=\Gamma_z(x_1,...,x_m).$$

In [5], Y.J. Cho, M. Matić and J.E. Pečarić proved the following inequality of Kurepa:

Theorem D. Let X be a 2-inner product space and $x_1,\ldots,x_m,y_1,\ldots y_m,z$ are vectors in X such that $\{x_1,...,x_m\}$, $\{y_1,...,y_m)$ are two sets of linearly independent and $z\notin V(x_1,...,x_m,y_1,\cdots,y_m))$. Then we have the inequality

$$|\tilde\Gamma_z(x_1,y_1,...,x_m,y_m)|^2\le\Gamma_z(x_1,...,x_m)\Gamma_z(y_1,...,y_m). \tag{2.1}$$

The equality occurs in (2.1) if and only if $V(x_1,\cdots,x_m)$ spans the same subspace as $V(y_1,\cdots,y_m)$ does. In the case, the inequality (2.1) is said to be *Kurepa's inequality* in 2-inner product spaces.

In this section, we shall give some inequality of Aczél type inequality for Gramians which generalize the results of Kurepa's inequality (2.1).

Theorem 2.1. Let X be a 2-inner product space and $a,b,c\in R$ satisfy the following condition:

$$a,c>0\quad\text{and}\quad b^2\ge ac.$$

Then for all $x_i, y_i \in X (i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ with

$$a \geq \Gamma_z(x_1, ..., x_m) \quad \text{or} \quad c \geq \Gamma_z(y_1, ..., y_m),$$

we have the inequality

$$[a - \Gamma_z(x_1, ..., x_m)][c - \Gamma_z(y_1, ..., y_m)] \leq [b - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)]^2. \tag{2.2}$$

Proof. Suppose that $a \geq \Gamma_z(x_1, ..., x_m)$ and consider the polynomial $p(t)$ defined by

$$p(t) = at^2 - 2bt + c, \quad \text{for all} \quad t \in R.$$

Since $a > 0$ and $b^2 \geq ac$, there exists $t_o \in R$ such that $p(t_o) = 0$. We put the polynomials

$$q(t) = p(t) - (\Gamma_z(x_1, ..., x_m)t^2 \mp 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)t + \Gamma_z(y_1, ..., y_m))$$

for all $t \in R$. A simple calculation gives

$$q(t) = [a - \Gamma_z(x_1, ..., x_m)]t^2 - 2[b \pm \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)]t + [c - \Gamma_z(y_1, ..., y_m)]$$

for all $t \in R$. Since

$$q(t_o) = -\left[\Gamma_z(x_1, ..., x_m)t_o^2 \mp 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)t_o + \Gamma_z(y_1, ..., y_m)\right] \leq 0$$

and from Kurepa's inequality (2.1) in Theorem C, we have

$$\Gamma_z(x_1, ..., x_m)t^2 \mp 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)t + \Gamma_z(y_1, ..., y_m) \geq 0$$

for all $t \in R$. Then we obtain that $q(t)$ has at least one solution in R, that is, the discriminant Δ of $q(t)$ must be non-negative. Therefore we have

$$0 \leq [b \pm \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)]^2 - [a - \Gamma_z(x_1, ..., x_m)][c - \Gamma_z(y_1, ..., y_m)]. \tag{2.3}$$

Next, suppose that $c \geq \Gamma_z(y_1, ..., y_m)$. Then we can be proved similarly by considering polynomials:

$$p_1(t) = ct^2 - 2bt + a \quad \text{for all} \quad t \in R$$

and

$$q_1(t) = p_1(t) - (\Gamma_z(y_1, ..., y_m)t^2 \pm 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)t + \Gamma_z(x_1, ..., x_m)).$$

This completes the proof. □

Corollary 2.2. Let X be a 2-inner product space and $M_1, M_2 \in R$. Then for all $x_i, y_i \in X (i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ such that

$$\Gamma_z(x_1, ..., x_m) \le |M_1| \quad \text{or} \quad \Gamma_z(y_1, ..., y_m) \le |M_2|$$

we have the inequality:

$$\begin{aligned} &[M_1^2 - \Gamma_z(x_1, ..., x_m)][M_2^2 - \Gamma_z(y_1, ..., y_m)] \\ &\le [M_1 M_2 - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)]^2. \end{aligned} \tag{2.4}$$

Proof. Note that $(m^2 - n^2)(p^2 - q^2) \le (mp - nq)^2$ for every $m, n, p, q \in R$. Using the above inequality and Kurepa's inequality in the 2-inner product space, then the desired results follows. This completes the proof. □

Corollary 2.3. Let $a, b, c, x_i, y_i \, (i = 1, ..., m), z$ be as in Theorem 2.1. Then we have the inequality:

$$\begin{aligned} 0 &\le \Gamma_z(x_1, ..., x_m)\Gamma_z(y_1, ..., y_m) - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)^2 \\ &\le b^2 - ac + a\Gamma_z(y_1, ..., y_m) + c\Gamma_z(x_1, ..., x_m) \\ &\quad - 2b\,\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m). \end{aligned} \tag{2.5}$$

Proof. The first inequality in (2.5) follows from the Kurepa's inequality (2.1), while the second we can be just rewritten as the inequality (2.2). This completes the proof. □

Corollary 2.4. Let X be a 2-inner product space and $M > 0$. If $x_i, y_i, z \in X$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ are such that

$$\Gamma_z(x_1, ..., x_m) \le M^2 \quad \text{or} \quad \Gamma_z(y_1, ..., y_m) \le M^2,$$

then we have the inequality:

$$\begin{aligned} 0 &\le \Gamma_z(x_1, ..., x_m)\Gamma_z(y_1, ..., y_m) - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)^2 \\ &\le M^2[\Gamma_z(x_1, ..., x_m) - 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m) + \Gamma_z(y_1, ..., y_m)]. \end{aligned} \tag{2.6}$$

Proof. We apply Corollary 2.3 with $a = b = c = M$ and the inequality (2.5) reduces to the inequality (2.6). This completes the proof. □

Next, we shall give the similar result of Theorem 2.1 can also be stated:

Theorem 2.5. Let X be a 2-inner product space and $\alpha, \beta, \gamma \in R$ satisfy the following condition:

$$\alpha, \gamma > 0 \quad \text{and} \quad \beta^2 \ge \alpha\gamma.$$

Then for all $x_i, y_i \in X(i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ with

$$\alpha \geq \Gamma_z(x_1, ..., x_m)^{1/2} \quad \text{or} \quad \gamma \geq \Gamma_z(y_1, ..., y_m)^{1/2},$$

we have the inequality:

$$\begin{aligned} [\alpha - &\Gamma_z(x_1, ..., x_m)^{1/2}][\gamma - \Gamma_z(y_1, ..., y_m)^{1/2}] \\ &\leq [\beta - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)^{1/2}]^2. \end{aligned} \tag{2.7}$$

Proof. Consider the polynomial

$$\begin{aligned} \tilde{q}(t) = \tilde{p}(t) - (\Gamma_z(x_1, ..., x_m)^{1/2} t^2 \mp 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)t \\ + \Gamma_z(y_1, ..., y_m)^{1/2}) \end{aligned}$$

where $\tilde{p}(t) = at^2 - 2bt + c$ for all $t \in R$. By a similar argument to that in proof of Theorem 2.1, we obtain the inequality (2.7). This completes the proof. □

Using Theorem 2.5, we can obtain the following corollaries:

Corollary 2.6. Let $x_i, z \in X$ $(i = 1, 2, .., m)$, $\alpha, \beta, \gamma \in R$ be as in Theorem 2.5. Then we have the inequality:

$$\begin{aligned} 0 &\leq \Gamma_z(x_1, ..., x_m)^{1/2} \Gamma_z(y_1, ..., y_m)^{1/2} - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m) \\ &\leq \beta^2 - \alpha\gamma + \alpha\Gamma_z(y_1, ..., y_m)^{1/2} + \gamma\Gamma_z(x_1, ..., x_m)^{1/2} \\ &\quad - 2\beta\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m). \end{aligned} \tag{2.8}$$

Corollary 2.7. Let X be a 2-inner product space and $M > 0$. If $x_i, y_i, z \in X$ are such that $\Gamma_z(x_1, ..., x_m)^{1/2} \leq M$ or $\Gamma_z(y_1, ..., y_m)^{1/2} \leq M$, then we have the inequality:

$$\begin{aligned} 0 &\leq \Gamma_z(x_1, ..., x_m)^{1/2} \Gamma_z(y_1, ..., y_m)^{1/2} - \tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m) \\ &\leq M[\Gamma_z(x_1, ..., x_m)^{1/2} + \Gamma_z(y_1, ..., y_m)^{1/2} \\ &\quad - 2\tilde{\Gamma}_z(x_1, y_1, ..., x_m, y_m)^{1/2}]. \end{aligned} \tag{2.9}$$

Remark 1. Using the inequality (1.6)

$$\Gamma_z(x_1, \ldots, x_k, x_{k+1} \ldots, x_m) \leq \Gamma_z(x_1, \ldots, x_k)\Gamma_z(x_{k+1}, \ldots, x_m)$$

and Corollary 2.3 and 2.4, we have the following:

(i) If $a, b, c \in R$ with $a, c > 0$ and $b^2 \geq ac > 0$, then for all $a \geq \Gamma_z(x_1, ..., x_k)^2$ or $c \geq \Gamma_z(x_{k+1}, ..., x_m)^2$, we have the inequality

$$[a - \Gamma_z(x_1, ..., x_k)^2][c - \Gamma_z(x_{k+1}, ..., x_m)^2] \leq [b - \Gamma_z(x_1, ..., x_m)]^2$$

from which we easily obtains

$$\begin{aligned} 0 &\le \Gamma_z(x_1,...,x_k)^2\Gamma_z(x_{k+1},...,x_m)^2 - \Gamma_z(x_1,...,x_m)^2 \\ &\le b^2 - ac + a\Gamma_z(x_{k+1},...,x_m)^2 \\ &\quad + c\Gamma_z(x_1,...,x_k)^2 \pm 2b\,\Gamma_z(x_1,...,x_m), \end{aligned} \tag{2.10}$$

(ii) Let $M > 0$. If $\Gamma_z(x_1,...,x_k) \le M$ or $\Gamma_z(x_{k+1},...,x_m) \le M$, then

$$\begin{aligned} 0 &\le \Gamma_z(x_1,...,x_k)^2\,\Gamma_z(x_{k+1},...,x_m)^2 - \Gamma_z(x_1,...,x_m)^2 \\ &\le M^2[\Gamma_z(x_1,...,x_k)^2 + \Gamma_z(x_{k+1},...,x_m)^2 - 2\,\Gamma_z(x_1,...,x_m)]. \end{aligned} \tag{2.11}$$

Remark 2. Next, from the inequality (1.6) and Corollary 2.6 and 2.7, we obtain the following:

(i) If $\alpha, \beta, \gamma \in R$ with $\alpha, \gamma > 0$ and $\beta^2 \le \alpha\gamma$, then for all $x_i \in X$ $(i = 1, ..., m)$ with $\alpha \ge \Gamma_z(x_1,...,x_k)$ or $\gamma \ge \Gamma_z(x_{k+1},...,x_m)$, we have the inequality

$$[\alpha - \Gamma_z(x_1,...,x_k)][\gamma - \Gamma_z(x_{k+1},...,x_m)] \le [\beta - \Gamma_z(x_1,...,x_m)^{1/2}]^2,$$

which gives

$$\begin{aligned} 0 &\le \Gamma_z(x_1,...,x_k)\Gamma_z(x_{k+1},...,x_m) - \Gamma_z(x_1,...,x_m) \\ &\le \beta^2 - \alpha\gamma + \alpha\Gamma_z(x_{k+1},...,x_m) \\ &\quad + \gamma\Gamma_z(x_1,...,x_k) \pm 2\beta\Gamma_z(x_1,...,x_m)^{1/2}, \end{aligned} \tag{2.12}$$

(ii) Let $M > 0$. If $\Gamma_z(x_1,...,x_k) \le M$ or $\Gamma_z(x_{k+1},...,x_m) \le M$, then

$$\begin{aligned} 0 &\le \Gamma_z(x_1,...,x_k)\Gamma_z(x_{k+1},...,x_m) - \Gamma_z(x_1,...,x_m) \\ &\le M[\Gamma_z(x_1,...,x_k) + \Gamma_z(x_{k+1},...,x_m) - 2\Gamma_z(x_1,...,x_m)^{1/2}]. \end{aligned} \tag{2.13}$$

3. Aczél's type inequality and Hadamard's inequality

Let X be a 2-inner product space, $x_1, x_2, \dots, x_m$ be the nonzero vectors in X and $z \in X$ such that $z \notin V(x_1,...,x_m)$. Then we have

$$\Gamma_z(x_1,...,x_m) \le \prod_{i=1}^{m} \|x_i, z\|^2. \tag{3.1}$$

For $m \ge 2$, the equality in (3.1) holds if and only if $(x_i, x_j|z) = \delta_{ij}\|x_i, z\|\|x_j, z\|$ for all $1 \le i, j \le m$. Then the inequality (3.1) is said to be *Hadamard's inequality* for the Gram determinant in 2-inner product spaces.

In this section, we shall give some inequality of Aczél's type inequality for Gramians which generalize the results of Hadamard's inequality (3.1).

Theorem 3.1. Let X be a 2-inner product space and $a, b, c \in R$ satisfying the following condition:

$$a, c > 0 \quad \text{and} \quad b^2 \geq ac.$$

Then for all $x_i \in X \,(i = 1, 2, ..., m,\ m \geq 2)$ and $z \notin V(x_1, ..., x_m)$ with

$$a \geq \prod_{i=1}^{k} \|x_i, z\|^4 \quad \text{or} \quad c \geq \prod_{i=k+1}^{m} \|x_i, z\|^4,$$

where $1 \leq k \leq m$, we have the inequality:

$$(b \pm \Gamma_z(x_1, ..., x_m))^2 \geq \left(a - \prod_{i=1}^{k} \|x_i, z\|^4\right)\left(c - \prod_{i=k+1}^{m} \|x_i, z\|^4\right). \tag{3.2}$$

Proof. Suppose that $a \geq \prod_{i=1}^{k} \|x_i, z\|^4$ and fixed $k \in \{1, ..., m\}$. Let us consider the polynomial

$$\psi(t) = at^2 - 2bt + c \quad \text{for all} \quad t \in R.$$

Then, since $a > 0$ and $b^2 \geq ac$, there exists $t_o \in R$ such that $\psi(t_o) = 0$. Now, put

$$\phi(t) = \psi(t) - \left[\left(\prod_{i=1}^{k} \|x_i, z\|^4\right)t^2 \mp 2b\Gamma(x_1, ..., x_m)t + \prod_{i=k+1}^{m} \|x_i, z\|^4\right]$$

for all $t \in R$. Then we have

$$\phi(t) = \left(a - \prod_{i=1}^{k} \|x_i, z\|^4\right)t^2 - 2(b \pm \Gamma_z(x_1, ..., x_m))t + (c - \prod_{i=k+1}^{m} \|x_i, z\|^4)$$

for all $t \in R$. By the Hadamard's inequality, we have

$$\Gamma_z(x_1, ..., x_m)^2 \leq \prod_{i=1}^{m} \|x_i, z\|^2 = \left(\prod_{i=1}^{k} \|x_i, z\|^4\right)\left(\prod_{i=k+1}^{m} \|x_i, z\|^4\right),$$

which gives

$$\left(\prod_{i=1}^{k} \|x_i, z\|^4\right)t^2 - 2\Gamma_z(x_1, ..., x_m)t + \prod_{i=k+1}^{m} \|x_i, z\|^4 \geq 0$$

for all $t \in R$. Since $\phi(t_o) \geq 0$, $\phi(t) = 0$ has at least one solution in R. By the discriminant we have

$$0 \leq \Big(b \pm \Gamma_z(x_1, ..., x_m)\Big)^2 - \Big(a - \prod_{i=1}^{k} \|x_i, z\|^4\Big)\Big(c - \prod_{i=k+1}^{m} \|x_i, z\|^4\Big). \tag{3.3}$$

Thus, by the inequalities (3.3), we have the inequality (3.2).

Next, suppose that $c \geq \prod_{i=k+1}^{m} \|x_i, z\|^4$. Then we have the same conclusion by considering polynomials

$$\psi_1(t) = ct^2 - 2bt + a, \quad \text{for all} \quad t \in R$$

and

$$\phi_1(t) = \psi_1(t) - \Big[\Big(\prod_{i=k+1}^{m} \|x_i, z\|^4\Big)t^2 \mp 2b\Gamma(x_1, ..., x_m)t + \prod_{i=1}^{k} \|x_i, z\|^4\Big].$$

This completes the proof. □

Using Theorem 3.1, we can state the following corollaries:

Corollary 3.2. Let a, b, c, x_i, z be as in Theorem 3.1. Then we have the inequality:

$$\begin{aligned} 0 &\leq \prod_{i=1}^{m} \|x_i, z\|^4 - \Gamma_z(x_1, ..., x_m)^2 \\ &\leq b^2 - ac + a\prod_{i=k+1}^{m} \|x_i, z\|^4 + c\prod_{i=1}^{k} \|x_i, z\|^4 \pm 2b\Gamma_z(x_1, ..., x_m). \end{aligned} \tag{3.4}$$

Corollary 3.3. Let X be a 2-inner product space and $M > 0$. If $x_i \in X (i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m)$ with $\prod_{i=1}^{k} \|x_i, z\|^2 \leq M$ or $\prod_{i=k+1}^{m} \|x_i, z\|^2 \leq M$, then we have the inequality:

$$\begin{aligned} 0 &\leq \prod_{i=1}^{m} \|x_i, z\|^4 - \Gamma_z(x_1, ..., x_m)^2 \\ &\leq M^2\Big(\prod_{i=1}^{k} \|x_i, z\|^4 - 2\Gamma_z(x_1, ..., x_m) + \prod_{i=k+1}^{m} \|x_i, z\|^4\Big). \end{aligned} \tag{3.5}$$

Theorem 3.4. Let X be a 2-inner product space and $\alpha, \beta, \gamma \in R$ satisfy the following condition:

$$\alpha, \gamma > 0 \quad \text{and} \quad \beta^2 \geq \alpha\gamma.$$

Then for all $x_i \in X$ $(i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m)$ with

$$\alpha \geq \prod_{i=1}^{k} \|x_i, z\|^2 \quad \text{or} \quad \gamma \geq \prod_{i=k+1}^{k} \|x_i, z\|^2$$

where $1 \le k \le m$, we have the inequality:

$$[\alpha - \prod_{i=1}^{k} \|x_i, z\|^2][\gamma - \prod_{i=k+1}^{m} \|x_i, z\|^2] \le [\beta - [\Gamma_z(x_1, ..., x_m)]^{1/2}]^2. \tag{3.6}$$

By the use of the above theorem, we can also give the following corollaries:

Corollary 3.5. Let $\alpha, \beta, \gamma \in R$, $x_i, z \in X\,(i = 1, ..., m)$ be as in Theorem 3.4. Then we have

$$\begin{aligned} 0 &\le \prod_{i=1}^{m} \|x_i, z\|^2 - \Gamma_z(x_1, ..., x_m) \\ &\le \beta^2 - \alpha\gamma + \alpha \prod_{i=k+1}^{m} \|x_i, z\|^2 + \gamma \prod_{i=1}^{k} \|x_i, z\|^2 - 2\beta\Gamma_z(x_1, ..., x_m)^{1/2}. \end{aligned} \tag{3.7}$$

Corollary 3.6. Let X be a 2-inner product space and $M > 0$. If $x_1, x_2, ..., x_m \in X$ and $z \notin V(x_1, ..., x_m)$ with

$$\prod_{i=1}^{k} \|x_i, z\|^2 \le M \qquad \text{or} \qquad \prod_{i=k+1}^{m} \|x_i, z\|^2 \le M,$$

then we have the inequality:

$$\begin{aligned} 0 &\le \prod_{i=1}^{m} \|x_i, z\|^2 - \Gamma_z(x_1, ..., x_m) \\ &\le M\left(\prod_{i=1}^{k} \|x_i, z\|^2 + \prod_{i=k+1}^{m} \|x_i, z\|^2 - 2\Gamma_z(x_1, ..., x_m)^{1/2}\right). \end{aligned} \tag{3.8}$$

References

[1] I. Brnettić, Y.J. Cho and S.S. Kim and J.E. Pečarić, *On Aczél's inequality in 2-inner product spaces and applications,* PanAmerican Math. J. 14(2004), 63-74.

[2] Y.J. Cho, C.S. Lin, S.S. Kim and A. Misiak, *Theory of 2-inner product spaces*, *Theory of 2-inner product spaces*, Nova Science Publishers Inc., New York 2001.

[3] Y.J. Cho, S.S. Dragomir, A. White and S.S. Kim, *Some inequalities in 2-inner product spaces*, Demonstratio Math., 32(1999), 485-493.

[4] Y.J. Cho, S.S. Dragomir, C.S. Lin, S.S. Kim and Y.-H. Kim, *Some Pečarić type inequalities in 2-inner product spaces and application*, Taiwaness J. Math., 12(2008), 357-372.

[5] Y.J. Cho, M. Matić and J.E. Pečarić, *On Gram's determinant in 2-inner product spaces*, J. Korean Math. Soc., 38(2001), 1125-1156.

[6] S.S. Dragomir, *A generalization of J. Aczél's inequality in inner product spaces*, Acta Math. Hungar. 65(1994), 141-148.
[7] S.S. Kim, S.S. Dragomir and Y.J. Cho, *On superadditivity and monotonicity of Gram's type inequality and applications*, Demonstratio Math., 36(2003), 807-817.
[8] S.S. Kim, S.S Dragomir and Y.J. Cho, *Some inequality for Gram's determinant and applications*, PanAmerican Math. J., 18(2008), 89-97.
[9] S. Kurepa, *On the Buniakowsky-Cauchy-Schwarz inequality*, Glasnik Mat. Ser III 1(21)(1966), 147-158.
[10] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, Norwell, MA, 1993.
[11] J.E. Pečarić, *On some classical inequalities in unitary spaces*, Mat. Bilten(Scopje), 16(1992), 63-72.

Seong Sik Kim
Department of Mathematics
Dongeui University
Pusan 614-714, Korea
e-mail:sskim@deu.ac.kr

Yeol Je Cho
Department of Mathematics Education and RINS
Gyeongsang National University
Chinju 660-701, Korea
e-mail:mathyjcho@yahoo.com

Sever S. Dragomir
School of Communication and informatics
Victoria University of Technology
PO Box 14428
Melbourne Vic. 8001, Australia
e-mail:sever.dragomir@vu.edu.au