ON ACZÉL’S TYPE INEQUALITY FOR GRAMIANS IN 2-INNER PRODUCT SPACES

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ABSTRACT. In this paper, some results related to Aczél’s type inequality for Gramians in terms of Kurepa’s and Hadamard’s inequality in 2-inner product spaces are given.

I. Introduction

In 1956, J. Aczél proved the following inequality, known in literature as Aczél’s inequality ([10]):

**Theorem A.** Let \( a = (a_1, a_2, \ldots, a_m) \) and \( b = (b_1, b_2, \ldots, b_m) \) be two sequences of real numbers such that

\[
a_1^2 - a_2^2 - \cdots - a_m^2 > 0 \quad \text{or} \quad b_1^2 - b_2^2 - \cdots - b_m^2 > 0.
\]

Then we have the inequality

\[
(a_1^2 - a_2^2 - \cdots - a_m^2)(b_1^2 - b_2^2 - \cdots - b_m^2) \leq (a_1b_1 - a_2b_2 - \cdots - a_mb_m)^2
\]

with the equality if and only if the sequences \( a \) and \( b \) are proportional.

In [9], S. Kurepa proved the following inequality of Aczél type which holds in Hilbert spaces:

**Theorem B.** Let \( X \) be a real Hilbert space and \( c \) a unit vector in \( X \). Suppose that \( a, b \in X \) are given vectors such that

\[
(u^2 - \|a_u\|^2)(v^2 - \|b_v\|^2) \geq 0
\]

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where \( u = (a, c), v = (b, c), a_o = a - uc \) and \( b_o = b - vc \). Then

\[
(1.2) \quad \left( u^2 - \|a_o\|^2 \right) \times \left( v^2 - \|b_o\|^2 \right) \geq (uv - (a_o, b_o))^2.
\]

If \( a \) and \( b \) are independent and strict inequality holds in (1.2), then strict inequality also holds in (1.2).

S.S. Dragomir ([6]) proved the generalization results of Aczél’s inequality:

**Theorem C.** Let \((X, (\cdot, \cdot))\) be an inner product space over the real and complex numbers field \( K \) and \( a, b, c \in R \) satisfy the following condition:

\[
a, c > 0 \quad \text{and} \quad b^2 \geq ac.
\]

Then, for all \( x, y \in X \) with \( a \geq \|x\|^2 \) or \( c \geq \|y\|^2 \), we have the inequality

\[
(1.3) \quad (a - \|x\|^2)(c - \|y\|^2) \leq \min \left\{ \left( b \pm \Re(x, y) \right)^2, \left( b \pm |\Re(x, y)| \right)^2, \left( b \pm \Im(x, y) \right)^2, \left( b \pm |\Im(x, y)| \right)^2, \left( b \pm |(x, y)| \right)^2 \right\}.
\]

Let \( X \) be a linear space of dimension greater than 1 and \((\cdot, \cdot|\cdot)\) be a real-valued function on \( X \times X \times X \) satisfying the following conditions:

\[
(2I_1) \quad (x, x|z) \geq 0,
\]

\[
(2I_2) \quad (x, x|z) = 0 \quad \text{if and only if} \quad x \quad \text{and} \quad z \quad \text{are linearly dependent},
\]

\[
(2I_3) \quad (x, y|z) = (y, x|z),
\]

\[
(2I_4) \quad (\alpha x, y|z) = \alpha(x, y|z) \quad \text{for any real number} \quad \alpha,
\]

\[
(2I_5) \quad (x + x', y|z) = (x, y|z) + (x', y|z).
\]

\((\cdot, \cdot|\cdot)\) is called a 2-inner product and \((X, (\cdot, \cdot|\cdot))\) a 2-inner product space ([2]).

Some basic properties of the 2-inner product \((\cdot, \cdot|\cdot)\) are as follows ([2]):

1. For all \( x, y, z \in X \),

\[
|(x, y|z)| \leq \sqrt{(x, x|z)(y, y|z)}.
\]

2. For all \( x, y \in X \), \((x, y|y) = 0\).

3. If \((X, (\cdot|\cdot))\) is an inner product space, then the 2-inner product \((\cdot, \cdot|\cdot)\) is defined on \( X \) by

\[
(x, y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)
\]

for all \( x, y, z \in X \).
Under the same assumptions over $X$, the real-valued function $\left\| \cdot, \cdot \right\|$ on $X \times X$ satisfying the following conditions:

\begin{enumerate}[(2N_1)]
  \item $\left\| x, y \right\| = 0$ if and only if $x$ and $y$ are linearly dependent,
  \item $\left\| x, y \right\| = \left\| y, x \right\|$, 
  \item $\left\| \alpha x, y \right\| = |\alpha| \left\| x, y \right\|$ for all real number $\alpha$,
  \item $\left\| x, y + z \right\| \leq \left\| x, y \right\| + \left\| x, z \right\|$. 
\end{enumerate}

$\left\| \cdot, \cdot \right\|$ is called a 2-norm on $X$ and $(X, \left\| \cdot, \cdot \right\|)$ a linear 2-normed space ([8]). Some of the basic properties of the 2-norms are that they are non-negative and $\left\| x, y + \alpha x \right\| = \left\| x, y \right\|$ for every $x, y$ in $X$ and every real number $\alpha$.

For any nonzero $x_1, x_2, \ldots, x_n$ in $X$, let $V(x_1, x_2, \ldots, x_n)$ denote the subspace of $X$ generated by $x_1, x_2, \ldots, x_n$. Whenever the notation $V(x_1, x_2, \ldots, x_n)$ is used, by it will understood $x_1, x_2, \ldots, x_n$ to be linearly independent.

Let $(X, (\cdot, \cdot))$ be a 2-inner product space. For any elements $x_1, \ldots, x_m, z \in X$ we define the Gram’s matrix $G_z(x_1, \ldots, x_m)$ of $x_1, \ldots, x_m$ with respect to $z$ by

$$G_z(x_1, \ldots, x_m) = \begin{pmatrix}
  (x_1, x_1|z) & \cdots & (x_1, x_m|z) \\
  \vdots & \ddots & \vdots \\
  (x_m, x_1|z) & \cdots & (x_m, x_m|z)
\end{pmatrix}$$

and Gram’s determinant $\Gamma_z(x_1, \ldots, x_m)$ of $x_1, \ldots, x_m$ with respect to $z$ by

$$\Gamma_z(x_1, \ldots, x_m) = \det G_z(x_1, \ldots, x_m)$$

$$= \det \begin{pmatrix}
  (x_1, x_1|z) & \cdots & (x_1, x_m|z) \\
  \vdots & \ddots & \vdots \\
  (x_m, x_1|z) & \cdots & (x_m, x_m|z)
\end{pmatrix}.$$ 

Then we have the following inequality:

(1.4) \[ \Gamma_z(x_1, \ldots, x_m) \geq 0. \]

The equality holds in (1.4) if and only if $x_1, \ldots, x_m, z$ are linearly dependent. The inequality (1.4) is said to be Gram’s inequality in 2-inner product spaces.

In the inequality (1.4), if $n = 2$, then

\[(x_1, x_1|z)(x_2, x_2|z) - |(x_1, x_2|z)|^2 \geq 0\]

with the equality if and only if $x_1, x_2, z$ are linearly dependent. This inequality can be regards as a generalization of Cauchy-Buniakowski-Schwarz’s inequality in 2-inner product spaces.

Some inequalities which involve Gram’s determinants are given ([5]):

(1.5) \[ \frac{\Gamma_z(x_1, \ldots, x_m)}{\Gamma_z(x_1, \ldots, x_k)} \leq \frac{\Gamma_z(x_2, \ldots, x_m)}{\Gamma_z(x_2, \ldots, x_k)} \leq \cdots \leq \frac{\Gamma_z(x_{k+1}, \ldots, x_m)}{\Gamma_z(x_{k+1}, \ldots, x_k)} \]
\[ \Gamma_z(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) \leq \Gamma_z(x_1, \ldots, x_k) \Gamma_z(x_{k+1}, \ldots, x_m), \]

\[ \Gamma_z(x_1 + y_1, x_2, \ldots, x_m)^{1/2} \leq \Gamma_z(x_1, x_2, \ldots, x_m)^{1/2} + \Gamma_z(y_1, x_2, \ldots, x_m)^{1/2}. \]

For further details on some properties of Aczél’s and Gram’s inequalities of 2-inner product spaces are given by [1], [3] - [5], [7] - [9].

In this paper, some results related to Aczél’s type inequality for Gramians in terms of Kurepa’s and Hadamard’s inequality, which generalize and extend the corresponding results of S.S. Dragomir [6] to 2-inner product spaces are given.

II. Aczél’s type inequality and Kurepa’s inequality

Let \( X \) be a 2-inner product space and \( x_1, \ldots, x_m, y_1, \ldots, y_m, z \in X \). We can define the determinant \( \tilde{\Gamma}_z(x_1, y_1, \ldots, x_m, y_m) \) by

\[ \tilde{\Gamma}_z(x_1, y_1, \ldots, x_m, y_m) = \det \begin{pmatrix} (x_1, y_1 | z) & \ldots & (x_1, y_m | z) \\ \vdots & \ddots & \vdots \\ (x_m, y_1 | z) & \ldots & (x_m, y_m | z) \end{pmatrix} \]

for \( z \notin V(x_1, \ldots, x_m, y_1, \ldots, y_m) \). Note that if \( y_1 = x_1, \ldots, y_m = x_m \), then

\[ \tilde{\Gamma}_z(x_1, y_1, \ldots, x_m, y_m) = \Gamma_z(x_1, \ldots, x_m). \]

In [5], Y.J. Cho, M. Matić and J.E. Pečarić proved the following inequality of Kurepa:

**Theorem D.** Let \( X \) be a 2-inner product space and \( x_1, \ldots, x_m, y_1, \ldots, y_m, z \) are vectors in \( X \) such that \( \{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\} \) are two sets of linearly independent and \( z \notin V(x_1, \ldots, x_m, y_1, \ldots, y_m) \). Then we have the inequality

\[ |\tilde{\Gamma}_z(x_1, y_1, \ldots, x_m, y_m)|^2 \leq \Gamma_z(x_1, \ldots, x_m) \Gamma_z(y_1, \ldots, y_m). \]

The equality occurs in (2.1) if and only if \( V(x_1, \ldots, x_m) \) spans the same subspace as \( V(y_1, \ldots, y_m) \) does. In the case, the inequality (2.1) is said to be **Kurepa’s inequality** in 2-inner product spaces.

In this section, we shall give some inequality of Aczél type inequality for Gramians which generalize the results of Kurepa’s inequality (2.1).

**Theorem 2.1.** Let \( X \) be a 2-inner product space and \( a, b, c \in R \) satisfy the following condition:

\[ a, c > 0 \quad \text{and} \quad b^2 \geq ac. \]
Then for all $x_i, y_i \in X (i = 1, \ldots, m)$ and $z \notin V (x_1, \ldots, x_m, y_1, \ldots, y_m)$ with

$$a \geq \Gamma_z (x_1, \ldots, x_m) \quad \text{or} \quad c \geq \Gamma_z (y_1, \ldots, y_m),$$

we have the inequality

$$[a - \Gamma_z (x_1, \ldots, x_m)] [c - \Gamma_z (y_1, \ldots, y_m)] \leq [b - \Gamma_z (x_1, y_1, \ldots, x_m, y_m)]^2. \quad (2.2)$$

**Proof.** Suppose that $a \geq \Gamma_z (x_1, \ldots, x_m)$ and consider the polynomial $p(t)$ defined by

$$p(t) = at^2 - 2bt + c, \quad \text{for all} \quad t \in R.$$

Since $a > 0$ and $b^2 \geq ac$, there exists $t_o \in R$ such that $p(t_o) = 0$. We put the polynomials

$$q(t) = p(t) - (\Gamma_z (x_1, \ldots, x_m)t^2 + 2\Gamma_z (x_1, y_1, \ldots, x_m, y_m)t + \Gamma_z (y_1, \ldots, y_m))$$

for all $t \in R$. A simple calculation gives

$$q(t) = [a - \Gamma_z (x_1, \ldots, x_m)]t^2 - 2[b \pm \Gamma_z (x_1, y_1, \ldots, x_m, y_m)]t + [c - \Gamma_z (y_1, \ldots, y_m)]$$

for all $t \in R$. Since

$$q(t_o) = -\left[\Gamma_z (x_1, \ldots, x_m)t_o^2 + 2\Gamma_z (x_1, y_1, \ldots, x_m, y_m)t_o + \Gamma_z (y_1, \ldots, y_m)\right] \leq 0$$

and from Kurepa’s inequality (2.1) in Theorem C, we have

$$\Gamma_z (x_1, \ldots, x_m)t^2 + 2\Gamma_z (x_1, y_1, \ldots, x_m, y_m)t + \Gamma_z (y_1, \ldots, y_m) \geq 0$$

for all $t \in R$. Then we obtain that $q(t)$ has at least one solution in $R$, that is, the discriminant $\Delta$ of $q(t)$ must be non-negative. Therefore we have

$$0 \leq [b \pm \Gamma_z (x_1, y_1, \ldots, x_m, y_m)]^2 - [a - \Gamma_z (x_1, \ldots, x_m)][c - \Gamma_z (y_1, \ldots, y_m)]. \quad (2.3)$$

Next, suppose that $c \geq \Gamma_z (y_1, \ldots, y_m)$. Then we can be proved similarly by considering polynomials:

$$p_1(t) = at^2 - 2bt + a, \quad \text{for all} \quad t \in R$$

and

$$q_1(t) = p_1(t) - (\Gamma_z (y_1, \ldots, y_m)t^2 + 2\Gamma_z (x_1, y_1, \ldots, x_m, y_m)t + \Gamma_z (x_1, \ldots, x_m)).$$

This completes the proof. □
Corollary 2.2. Let $X$ be a 2-inner product space and $M_1, M_2 \in R$. Then for all $x_i, y_i \in X (i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ such that

$$\Gamma_z(x_1, ..., x_m) \leq |M_1| \text{ or } \Gamma_z(y_1, ..., y_m) \leq |M_2|$$

we have the inequality:

$$[M_1^2 - \Gamma_z(x_1, ..., x_m)][M_2^2 - \Gamma_z(y_1, ..., y_m)] \leq [M_1M_2 - \Gamma_z(x_1, y_1, ..., x_m, y_m)]^2. \tag{2.4}$$

**Proof.** Note that $(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2$ for every $m, n, p, q \in R$. Using the above inequality and Kurepa’s inequality in the 2-inner product space, then the desired results follows. This completes the proof. □

Corollary 2.3. Let $a, b, c, x_i, y_i (i = 1, ..., m), z$ be as in Theorem 2.1. Then we have the inequality:

$$0 \leq \Gamma_z(x_1, ..., x_m)\Gamma_z(y_1, ..., y_m) - \Gamma_z(x_1, y_1, ..., x_m, y_m)^2 \leq b^2 - ac + a\Gamma_z(y_1, ..., y_m) + c\Gamma_z(x_1, ..., x_m) - 2b\Gamma_z(x_1, y_1, ..., x_m, y_m). \tag{2.5}$$

**Proof.** The first inequality in (2.5) follows from the Kurepa’s inequality (2.1), while the second we can be just rewritten as the inequality (2.2). This completes the proof. □

Corollary 2.4. Let $X$ be a 2-inner product space and $M > 0$. If $x_i, y_i, z \in X$ and $z \notin V(x_1, ..., x_m, y_1, ..., y_m)$ are such that

$$\Gamma_z(x_1, ..., x_m) \leq M^2 \text{ or } \Gamma_z(y_1, ..., y_m) \leq M^2,$$

then we have the inequality:

$$0 \leq \Gamma_z(x_1, ..., x_m)\Gamma_z(y_1, ..., y_m) - \Gamma_z(x_1, y_1, ..., x_m, y_m)^2 \leq M^2[\Gamma_z(x_1, ..., x_m) - 2\Gamma_z(x_1, y_1, ..., x_m, y_m) + \Gamma_z(y_1, ..., y_m)]. \tag{2.6}$$

**Proof.** We apply Corollary 2.3 with $a = b = c = M$ and the inequality (2.5) reduces to the inequality (2.6). This completes the proof. □

Next, we shall give the similar result of Theorem 2.1 can also be stated:

**Theorem 2.5.** Let $X$ be a 2-inner product space and $\alpha, \beta, \gamma \in R$ satisfy the following condition:

$$\alpha, \gamma > 0 \text{ and } \beta^2 \geq \alpha \gamma.$$
Then for all $x_i, y_i \in X (i = 1, \ldots, m)$ and $z \notin V (x_1, \ldots, x_m, y_1, \ldots, y_m)$ with
\[
\alpha \geq \Gamma_z (x_1, \ldots, x_m)^{1/2} \quad \text{or} \quad \gamma \geq \Gamma_z (y_1, \ldots, y_m)^{1/2},
\]
we have the inequality:
\begin{equation}
\tag{2.7}
[\alpha - \Gamma_z (x_1, \ldots, x_m)^{1/2}] [\gamma - \Gamma_z (y_1, \ldots, y_m)^{1/2}] \\
\leq [\beta - \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m)^{1/2}]^2.
\end{equation}

Proof. Consider the polynomial
\[
\tilde{q}(t) = \tilde{p}(t) - (\Gamma_z (x_1, \ldots, x_m)^{1/2} t^2 + 2 \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m) t \\
+ \Gamma_z (y_1, \ldots, y_m)^{1/2})
\]
where $\tilde{p}(t) = at^2 - 2bt + c$ for all $t \in R$. By a similar argument to that in proof of Theorem 2.1, we obtain the inequality (2.7). This completes the proof. \qed

Using Theorem 2.5, we can obtain the following corollaries:

**Corollary 2.6.** Let $x_i, z \in X (i = 1, 2, \ldots, m)$, $\alpha, \beta, \gamma \in R$ be as in Theorem 2.5. Then we have the inequality:
\begin{equation}
\tag{2.8}
0 \leq \Gamma_z (x_1, \ldots, x_m)^{1/2} \Gamma_z (y_1, \ldots, y_m)^{1/2} - \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m) \\
\leq \beta^2 - \alpha \gamma + \alpha \Gamma_z (y_1, \ldots, y_m)^{1/2} + \gamma \Gamma_z (x_1, \ldots, x_m)^{1/2} \\
- 2 \beta \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m).
\end{equation}

**Corollary 2.7.** Let $X$ be a 2-inner product space and $M > 0$. If $x_i, y_i, z \in X$ are such that $\Gamma_z (x_1, \ldots, x_m)^{1/2} \leq M$ or $\Gamma_z (y_1, \ldots, y_m)^{1/2} \leq M$, then we have the inequality:
\begin{equation}
\tag{2.9}
0 \leq \Gamma_z (x_1, \ldots, x_m)^{1/2} \Gamma_z (y_1, \ldots, y_m)^{1/2} - \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m) \\
\leq M [\Gamma_z (x_1, \ldots, x_m)^{1/2} + \Gamma_z (y_1, \ldots, y_m)^{1/2}] \\
- 2 \tilde{\Gamma}_z (x_1, y_1, \ldots, x_m, y_m)^{1/2}].
\end{equation}

**Remark 1.** Using the inequality (1.6)
\[
\Gamma_z (x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) \leq \Gamma_z (x_1, \ldots, x_k) \Gamma_z (x_{k+1}, \ldots, x_m)
\]
and Corollary 2.3 and 2.4, we have the following:

(i) If $a, b, c \in R$ with $a, c > 0$ and $b^2 \geq ac > 0$, then for all $a \geq \Gamma_z (x_1, \ldots, x_k)^2$ or $c \geq \Gamma_z (x_{k+1}, \ldots, x_m)^2$, we have the inequality
\[
[a - \Gamma_z (x_1, \ldots, x_k)^2] [c - \Gamma_z (x_{k+1}, \ldots, x_m)^2] \leq [b - \Gamma_z (x_1, \ldots, x_m)]^2.
\]
from which we easily obtains
\[
0 \leq \Gamma_z(x_1, \ldots, x_k)^2 \Gamma_z(x_{k+1}, \ldots, x_m)^2 - \Gamma_z(x_1, \ldots, x_m)^2
\]
(2.10)
\[
\leq b^2 - ac + a\Gamma_z(x_{k+1}, \ldots, x_m)^2 + c\Gamma_z(x_1, \ldots, x_k)^2 + 2b\Gamma_z(x_1, \ldots, x_m),
\]

(ii) Let \( M > 0 \). If \( \Gamma_z(x_1, \ldots, x_k) \leq M \) or \( \Gamma_z(x_{k+1}, \ldots, x_m) \leq M \), then
\[
0 \leq \Gamma_z(x_1, \ldots, x_k)^2 \Gamma_z(x_{k+1}, \ldots, x_m)^2 - \Gamma_z(x_1, \ldots, x_m)^2
\]
(2.11)
\[
\leq M^2[\Gamma_z(x_1, \ldots, x_k)^2 + \Gamma_z(x_{k+1}, \ldots, x_m)^2 - 2\Gamma_z(x_1, \ldots, x_m)].
\]

Remark 2. Next, from the inequality (1.6) and Corollary 2.6 and 2.7, we obtain the following:
(i) If \( \alpha, \beta, \gamma \in R \) with \( \alpha, \gamma > 0 \) and \( \beta^2 \leq \alpha\gamma \), then for all \( x_i \in X \) \((i = 1, \ldots, m)\) with \( \alpha \geq \Gamma_z(x_1, \ldots, x_k) \) or \( \gamma \geq \Gamma_z(x_{k+1}, \ldots, x_m) \), we have the inequality
\[
[\alpha - \Gamma_z(x_1, \ldots, x_k)][\gamma - \Gamma_z(x_{k+1}, \ldots, x_m)] \leq [\beta - \Gamma_z(x_1, \ldots, x_m)^{1/2}]^2,
\]
which gives
\[
0 \leq \Gamma_z(x_1, \ldots, x_k) \Gamma_z(x_{k+1}, \ldots, x_m) - \Gamma_z(x_1, \ldots, x_m)
\]
(2.12)
\[
\leq \beta^2 - \alpha\gamma + \alpha\Gamma_z(x_{k+1}, \ldots, x_m)
\]
\[
+ \gamma\Gamma_z(x_1, \ldots, x_k) + 2\beta\Gamma_z(x_1, \ldots, x_m)^{1/2},
\]

(ii) Let \( M > 0 \). If \( \Gamma_z(x_1, \ldots, x_k) \leq M \) or \( \Gamma_z(x_{k+1}, \ldots, x_m) \leq M \), then
\[
0 \leq \Gamma_z(x_1, \ldots, x_k) \Gamma_z(x_{k+1}, \ldots, x_m) - \Gamma_z(x_1, \ldots, x_m)
\]
(2.13)
\[
\leq M[\Gamma_z(x_1, \ldots, x_k) + \Gamma_z(x_{k+1}, \ldots, x_m) - 2\Gamma_z(x_1, \ldots, x_m)^{1/2}].
\]

3. Aczél’s type inequality and Hadamard’s inequality

Let \( X \) be a 2-inner product space, \( x_1, x_2, \ldots, x_m \) be the nonzero vectors in \( X \) and \( z \in X \) such that \( z \notin V(x_1, \ldots, x_m) \). Then we have
\[
\Gamma_z(x_1, \ldots, x_m) \leq \prod_{i=1}^{m} \|x_i, z\|^2.
\]
(3.1)

For \( m \geq 2 \), the equality in (3.1) holds if and only if \( (x_i, x_j|z) = \delta_{ij}\|x_i, z\|\|x_j, z\| \) for all \( 1 \leq i, j \leq m \). Then the inequality (3.1) is said to be Hadamard’s inequality for the Gram determinant in 2-inner product spaces.
In this section, we shall give some inequality of Aczél’s type inequality for Gramians which generalize the results of Hadamard’s inequality (3.1).

**Theorem 3.1.** Let $X$ be a 2-inner product space and $a, b, c \in R$ satisfying the following condition: 

$$a, c > 0 \quad \text{and} \quad b^2 \geq ac.$$ 

Then for all $x_i \in X \ (i = 1, 2, ..., m, \ m \geq 2)$ and $z \notin V(x_1, ..., x_m)$ with

$$a \geq \prod_{i=1}^{k} \|x_i, z\|^4 \quad \text{or} \quad c \geq \prod_{i=k+1}^{m} \|x_i, z\|^4,$$

where $1 \leq k \leq m$, we have the inequality:

$$(3.2) \quad (b \pm \Gamma_z(x_1, ..., x_m))^2 \geq \left( a - \prod_{i=1}^{k} \|x_i, z\|^4 \right) \left( c - \prod_{i=k+1}^{m} \|x_i, z\|^4 \right).$$

**Proof.** Suppose that $a \geq \prod_{i=1}^{k} \|x_i, z\|^4$ and fixed $k \in \{1, ..., m\}$. Let us consider the polynomial

$$\psi(t) = at^2 - 2bt + c \quad \text{for all} \quad t \in R.$$ 

Then, since $a > 0$ and $b^2 \geq ac$, there exists $t_o \in R$ such that $\psi(t_o) = 0$. Now, put

$$\phi(t) = \psi(t) - \left[ \left( \prod_{i=1}^{k} \|x_i, z\|^4 \right) t^2 \mp 2b \Gamma_z(x_1, ..., x_m) t + \prod_{i=k+1}^{m} \|x_i, z\|^4 \right]$$

for all $t \in R$. Then we have

$$\phi(t) = \left( a - \prod_{i=1}^{k} \|x_i, z\|^4 \right) t^2 \mp 2(b \pm \Gamma_z(x_1, ..., x_m)) t + \left( c - \prod_{i=k+1}^{m} \|x_i, z\|^4 \right)$$

for all $t \in R$. By the Hadamard’s inequality, we have

$$\Gamma_z(x_1, ..., x_m)^2 \leq \prod_{i=1}^{m} \|x_i, z\|^2 = \left( \prod_{i=1}^{k} \|x_i, z\|^4 \right) \left( \prod_{i=k+1}^{m} \|x_i, z\|^4 \right),$$

which gives

$$\left( \prod_{i=1}^{k} \|x_i, z\|^4 \right) t^2 \mp 2\Gamma_z(x_1, ..., x_m) t + \prod_{i=k+1}^{m} \|x_i, z\|^4 \geq 0.$$
for all $t \in R$. Since $\phi(t_0) \geq 0$, $\phi(t) = 0$ has at least one solution in $R$. By the discriminant we have

$$0 \leq \left( b + \Gamma_z(x_1, ..., x_m) \right)^2 - \left( a - \prod_{i=1}^{k} \|x_i, z\|^4 \right) \left( c - \prod_{i=k+1}^{m} \|x_i, z\|^4 \right).$$

Thus, by the inequalities (3.3), we have the inequality (3.2).

Next, suppose that $c \geq \prod_{i=k+1}^{m} \|x_i, z\|^4$. Then we have the same conclusion by considering polynomials

$$\psi_1(t) = ct^2 - 2bt + a, \quad \text{for all } \quad t \in R$$

and

$$\phi_1(t) = \psi_1(t) - \left[ \left( \prod_{i=k+1}^{m} \|x_i, z\|^4 \right) t^2 + 2b\Gamma_z(x_1, ..., x_m)t + \prod_{i=1}^{k} \|x_i, z\|^4 \right].$$

This completes the proof. □

Using Theorem 3.1, we can state the following corollaries:

**Corollary 3.2.** Let $a, b, c, x_i, z$ be as in Theorem 3.1. Then we have the inequality:

$$0 \leq \prod_{i=1}^{m} \|x_i, z\|^4 - \Gamma_z(x_1, ..., x_m)^2$$

$$\leq b^2 - ac + a \prod_{i=k+1}^{m} \|x_i, z\|^4 + c \prod_{i=1}^{k} \|x_i, z\|^4 \pm 2b\Gamma_z(x_1, ..., x_m).$$

**Corollary 3.3.** Let $X$ be a 2-inner product space and $M > 0$. If $x_i \in X(i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m)$ with $\prod_{i=1}^{k} \|x_i, z\|^2 \leq M$ or $\prod_{i=k+1}^{m} \|x_i, z\|^2 \leq M$, then we have the inequality:

$$0 \leq \prod_{i=1}^{m} \|x_i, z\|^4 - \Gamma_z(x_1, ..., x_m)^2$$

$$\leq M^2 \left( \prod_{i=1}^{k} \|x_i, z\|^4 - 2\Gamma_z(x_1, ..., x_m) + \prod_{i=k+1}^{m} \|x_i, z\|^4 \right).$$

**Theorem 3.4.** Let $X$ be a 2-inner product space and $\alpha, \beta, \gamma \in R$ satisfy the following condition:

$$\alpha, \gamma > 0 \quad \text{and} \quad \beta^2 \geq \alpha \gamma.$$ 

Then for all $x_i \in X(i = 1, ..., m)$ and $z \notin V(x_1, ..., x_m)$ with

$$\alpha \geq \prod_{i=1}^{k} \|x_i, z\|^2 \quad \text{or} \quad \gamma \geq \prod_{i=k+1}^{m} \|x_i, z\|^2$$
where $1 \leq k \leq m$, we have the inequality:

\begin{equation}
(3.6) \quad \left[ \alpha - \prod_{i=1}^{k} \|x_i, z\|^2 \right] \left[ \gamma - \prod_{i=k+1}^{m} \|x_i, z\|^2 \right] \leq \left[ \beta - [\Gamma_z(x_1, ..., x_m)]^{1/2} \right]^2.
\end{equation}

By the use of the above theorem, we can also give the following corollaries:

**Corollary 3.5.** Let $\alpha, \beta, \gamma \in \mathbb{R}, x_i, z \in X$ ($i = 1, ..., m$) be as in Theorem 3.4. Then we have

\begin{equation}
0 \leq \prod_{i=1}^{m} \|x_i, z\|^2 - \Gamma_z(x_1, ..., x_m)
\end{equation}

\begin{equation}
\leq \beta^2 - \alpha \gamma + \alpha \prod_{i=k+1}^{m} \|x_i, z\|^2 + \gamma \prod_{i=1}^{k} \|x_i, z\|^2 - 2\beta \Gamma_z(x_1, ..., x_m)^{1/2}.
\end{equation}

**Corollary 3.6.** Let $X$ be a 2-inner product space and $M > 0$. If $x_1, x_2, ..., x_m \in X$ and $z \notin V(x_1, ..., x_m)$ with

\begin{equation}
\prod_{i=1}^{k} \|x_i, z\|^2 \leq M \quad \text{or} \quad \prod_{i=k+1}^{m} \|x_i, z\|^2 \leq M,
\end{equation}

then we have the inequality:

\begin{equation}
0 \leq \prod_{i=1}^{m} \|x_i, z\|^2 - \Gamma_z(x_1, ..., x_m)
\end{equation}

\begin{equation}
\leq M \left( \prod_{i=1}^{k} \|x_i, z\|^2 + \prod_{i=k+1}^{m} \|x_i, z\|^2 - 2\Gamma_z(x_1, ..., x_m)^{1/2} \right).
\end{equation}

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