ON THE SUPERADDITIVITY AND MONOTONICITY OF MAPPINGS ASSOCIATED WITH CAUCHY-SCHWARZ'S INEQUALITY IN 2-INNER PRODUCT SPACES II

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Abstract. Superadditivity and monotonicity of some mappings associated with refinements of Cauchy-Schwarz's inequality in 2-inner product spaces are given.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book ([2]). Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 and $(\cdot,\cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

(2I$_1$) $(x, x|z) \geq 0$,
  $(x, x|z) = 0$ if and only if x and z are linearly dependent,
(2I$_2$) $(x, x|z) = (z, z|x)$,
(2I$_3$) $(x, y|z) = (y, x|z)$,
(2I$_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any real number α,
(2I$_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(.,.|.)$ is called a *2-inner product* and $(X, (\cdot,\cdot|\cdot))$ is called a *2-inner product space*. Some basic properties of the 2-inner product $(\cdot,\cdot|\cdot)$ are as follows ([2]):

(1) For all $x, y, z \in X$,

$$|(x, y|z)| \leq \sqrt{(x, x|z)}\sqrt{(y, y|z)}.$$

(2) For all $x, y \in X, (x, y|y) = 0$ and $(x, y|0) = 0$.

(3) If $(X, (\cdot|\cdot))$ is an inner product space, then the 2-inner product $(\cdot,\cdot|\cdot)$ is defined on X by

$$(x, y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)$$

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for all $x, y, z \in X$.

Under the same assumptions over X, the real-valued function $\|\cdot,\cdot\|$ on $X \times X$ satisfying the following conditions:

(2N$_1$) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2N$_2$) $\|x, y\| = \|y, x\|$,
(2N$_3$) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all real number α,
(2N$_4$) $\|x, y + z\| \le \|x, y\| + \|x, z\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space*. Some of the basic properties of the 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for every x, y in X and every real number α ([4]).

Whenever a 2-inner product space $(X, (\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space on $(X, \|\cdot\|)$ with the norm defined by $\|x, z\| = \sqrt{(x, x|z)}$ for all $x, z \in X$ and for any nonzero $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n)$ denote the subspace of X generated by $x_1, x_2, ..., x_n$.

Let $(X, (\cdot,\cdot|\cdot))$ be a 2-inner product space. If $(e_i)_{1\le i\le n}$ are linearly independent vectors in X and, for a given $z \in X$, $(e_i, e_j|z) = \delta_{ij}$ for all $i, j \in \{1, ..., n\}$ where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1\le i\le n}$ is *z-orthonormal*), then the following inequality is the corresponding *Bessel's inequality* for the z-orthonormal family $(e_i)_{1\le i\le n}$ in X

$$\sum_{i=1}^{n} |(x, e_i|z)|^2 \le \|x, z\|^2 \tag{1.1}$$

for any $x \in X$. For more details on this inequality, see [1], [3], [5]-[6].

For a 2-inner product space $(X, (\cdot,\cdot|\cdot))$, Cauchy-Schwarz's inequality

$$|(x, y|z)| \le \|x, z\|\|y, z\| \quad \text{and} \quad |(x, y|z)|^2 \le \|x, z\|^2\|y, z\|^2, \tag{1.2}$$

a 2-dimensional analogue of Cauchy-Schwarz's inequality, holds ([2]). The following refinements of Cauchy-Schwarz's inequality in 2-inner product spaces has been obtains in [6]:

$$0 \le |(x, y|z)| \le |(x, y|z) - \sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)| + \sum_{i=1}^{n} |(x, e_i|z)(e_i, y|z)| \le \|x, z\|\|y, z\|, \tag{1.3}$$

$$\begin{aligned} 0 &\le \left(\sum_{i=1}^{n} |(x, e_i|z)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(e_i, y|z)|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)\right| \\ &\le \|x, z\|\|y, z\| - |(x, y|z)|, \end{aligned} \tag{1.4}$$

$$\begin{aligned} 0 &\le \left|(x, y|z) - \sum_{i=1}^{n}(x, e_i|z)(e_i, y|z)\right|^2 \\ &\le \left(\|x, z\|^2 - \sum_{i=1}^{n} |(x, e_i|z)|^2\right)\left(\|y, z\|^2 - \sum_{i=1}^{n} |(y, e_i|z)|^2\right) \\ &\le \left(\|x, z\|\|y, z\| - \sum_{i=1}^{n} |(x, e_i|z)(e_i, y|z)|\right)^2, \end{aligned} \tag{1.5}$$

$$0 \le |\sum_{i=1}^{n}(x, e_i|z)(y, e_i|z)|^2 \le \sum_{i=1}^{n}|(x, e_i|z)|^2 \sum_{i=1}^{n}|(y, e_i|z)|^2 \le \|x, z\|^2\|y, z\|^2 - |(x, y|z)|^2 \tag{1.6}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$ and $(e_i)_{1\le i\le n}$ be a family of z-orthonomal vectors in X.

In this paper, superadditivity and monotonicity of some index set mappings associated with the refinements of Cauchy-Schwarz's inequality in 2-inner product spaces are given.

2. Main Results

Fixed a family $(e_i)_{1\le i\le n}$ of z-orthonormal vectors in a 2-inner product space X and $P(N)$ be the class of all finite indices of N. We can consider the index set mappings $\gamma, \delta : P(N) \times X^3 \to R$ given by

$$\gamma(I, x, y, z) = \sum_{i\in I}|(x, e_i|z)|^2 \sum_{i\in I}|(y, e_i|z)|^2 - |\sum_{i\in I}(x, e_i|z)(e_i, y|z)|^2$$

and

$$\begin{aligned}\delta(I, x, y, z) = &\left(\|x, z\|^2 - \sum_{i\in I}|(x, e_i|z)|^2\right)\left(\|y, z\|^2 - \sum|(y, e_i|z)|^2\right)\\ &- \left|(x, y|z) - \sum_{i\in I}(x, e_i|z)(e_i, y|z)\right|^2\end{aligned}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$ and $I \in P(N)$.

Theorem 2.1. Let X be a 2-inner product space, $(e_i)_{1\le i\le n}$ be a z-orthonormal family of vectors in X and $P(N)$ be the class of all finite indices of N. Then we have

(i) For all $I, J \in P(N)$ with $I \cap J = \phi$

$$\begin{aligned}&\gamma(I \cup J, x, y, z) - \gamma(I, x, y, z) - \gamma(J, x, y, z)\\ &\quad\ge \left(\det\begin{bmatrix}(\sum_{i\in I}|(x, e_i|z)|^2)^{1/2} & (\sum_{i\in I}|(y, e_i|z)|^2)^{1/2}\\ (\sum_{j\in J}|(x, e_j|z)|^2)^{1/2} & (\sum_{j\in J}|(y, e_j|z)|^2)^{1/2}\end{bmatrix}\right)^2 \ge 0\end{aligned} \tag{2.1}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$, i.e., the mapping $\gamma(\cdot, x, y, z)$ is superadditive on $P(N)$.

(ii) For all $I, J \in P(N)$ with $I \supset J(\ne \phi)$ and $I \ne J$

$$\begin{aligned}&\gamma(I, x, y, z) - \gamma(J, x, y, z)\\ &\quad\ge \left(\det\begin{bmatrix}(\sum_{j\in J}|(x, e_j|z)|^2)^{1/2} & (\sum_{j\in J}|(y, e_j|z)|^2)^{1/2}\\ (\sum_{k\in I\setminus J}|(x, e_k|z)|^2)^{1/2} & (\sum_{k\in I\setminus J}|(y, e_k|z)|^2)^{1/2}\end{bmatrix}\right)^2 \ge 0\end{aligned} \tag{2.2}$$

for all $x, y, z \in X$ and $z \notin V(x, y)$, i.e., the mapping $\gamma(\cdot, x, y, z)$ is strong monotone nondecreasing on $P(N)$.

Proof. (i) Let $I, J \in P(N)$ with $I \cap J = \phi$. Then we have

$$\begin{aligned}
&\gamma(I \cup J, x, y, z) \\
&= \sum_{k\in I\cup J} |(x,e_k|z)|^2 \sum_{k\in I\cup J} |(y,e_k|z)|^2 - \left|\sum_{k\in I\cup J} (x,e_k|z)(e_k,y|z)\right|^2 \\
&= \left(\sum_{i\in I} |(x,e_i|z)|^2 + \sum_{j\in J} |(x,e_j|z)|^2\right)\left(\sum_{i\in I} |(y,e_i|z)|^2 + \sum_{j\in J} |(y,e_j|z)|^2\right) \\
&\qquad - \left|\sum_{i\in I}(x,e_i|z)(e_i,y|z) + \sum_{j\in J}(x,e_j|z)(e_j,y|z)\right|^2 \\
&= \sum_{i\in I} |(x,e_i|z)|^2 \sum_{i\in I} |(y,e_i|z)|^2 + \sum_{j\in J} |(x,e_j|z)|^2 \sum_{j\in J} |(y,e_j|z)|^2 \qquad (2.3)\\
&\qquad + \sum_{i\in I} |(x,e_i|z)|^2 \sum_{j\in J} |(y,e_j|z)|^2 + \sum_{j\in J} |(x,e_j|z)|^2 \sum_{i\in I} |(y,e_i|z)|^2 \\
&\qquad - |\sum_{i\in I}(x,e_i|z)(e_i,y|z)|^2 - |\sum_{j\in J}(x,e_j|z)(e_j,y|z)|^2 \\
&\qquad - 2\left[\sum_{i\in I}(x,e_i|z)(e_i,y|z)\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right] \\
&= \gamma(I,x,y,z) + \gamma(J,x,y,z) + k(I,J,x,y,z)
\end{aligned}$$

where

$$\begin{aligned}
&\kappa(I,J,x,y,z) \\
&= \sum_{i\in I} |(x,e_i|z)|^2 \sum_{j\in J} |(y,e_j|z)|^2 + \sum_{j\in J} |(x,e_j|z)|^2 \sum_{i\in I} |(y,e_i|z)|^2 \\
&\qquad - 2\left[\sum_{i\in I} |(x,e_i|z)(e_i,y|z)\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right]
\end{aligned}$$

for all $x, y, z \in X$ with $z \notin V(x,y)$. Then we have

$$\begin{aligned}
&\kappa(I,J,x,y,z) \\
&= \left(\det\begin{bmatrix} (\sum_{i\in I} |(x,e_i|z)|^2)^{1/2} & (\sum_{i\in I} |(y,e_i|z)|^2)^{1/2} \\ (\sum_{j\in J} |(x,e_j|z)|^2)^{1/2} & (\sum_{j\in J} |(y,e_j|z)|^2)^{1/2} \end{bmatrix}\right)^2 \\
&\quad + 2\Bigg\{\left(\sum_{i\in I} |(x,e_i|z)|^2 \sum_{i\in I} |(y,e_i|z)|^2\right)^{1/2}\left(\sum_{j\in J} |(x,e_j|z)|^2 \sum_{j\in J} |(y,e_j|z)|^2\right)^{1/2} \\
&\qquad - \left[\sum_{i\in I}(x,e_i|z)(e_i,y|z)\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right]\Bigg\}.
\end{aligned}$$

But, by Cauchy-Schwarz's inequality in a 2-inner product space, we have

$$\begin{aligned}
&\left(\sum_{i\in I} |(x,e_i|z)|^2 \sum_{i\in I} |(y,e_i|z)|^2\right)^{1/2}\left(\sum_{j\in J} |(x,e_j|z)|^2 \sum_{j\in J} |(y,e_j|z)|^2\right)^{1/2} \\
&\qquad \geq \left|\sum_{i\in I}(x,e_i|z)(e_i,y|z)\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right|
\end{aligned}$$

and so $\kappa(I,J,x,y,z)$ is nonnegative. Thus, we have the inequality (2.1).

(ii) Suppose that $I,J\in P(N)$ with $I\supset J(\neq\phi)$ and $I\neq J$. Then, by (i) we have

$$\begin{aligned}&\gamma((I\setminus J)\cup J,x,y,z)-\gamma(I\setminus J,x,y,z)-\gamma(J,x,y,z)\\&=\gamma(I,x,y,z)-\gamma(I\setminus J,x,y,z)-\gamma(J,x,y,z)\\&\geq\left(\det\begin{bmatrix}(\sum_{j\in J}|(x,e_j|z)|^2)^{1/2} & (\sum_{j\in J}|(y,e_j|z)|^2)^{1/2}\\(\sum_{k\in I\setminus J}|(x,e_k|z)|^2)^{1/2} & (\sum_{k\in I\setminus J}|(y,e_k|z)|^2)^{1/2}\end{bmatrix}\right)^2\end{aligned}$$

for all $x,y,z\in X$ with $z\notin V(x,y)$. Thus, we have the inequality (2.2). This completes the proof. □

Theorem 2.2. Let X be a 2-inner product space, $(e_i)_{1\leq i\leq n}$ be a z-orthonormal family of vectors in X and $P(N)$ be the class of all finite indices of N. Then we have the inequality:

$$\begin{aligned}\delta(I,x,y,z)&\geq\delta(I\cup J,x,y,z)+\gamma(J,x,y,z)\\&\quad+\left(\det\begin{bmatrix}\|x-\sum_{k\in I\cup J}(x,e_k|z)e_k,z\| & (\sum_{j\in J}|(x,e_j|z)|^2)^{1/2}\\\|y-\sum_{k\in I\cup J}(y,e_k|z)e_k,z\| & (\sum_{j\in J}|(y,e_j|z)|^2)^{1/2}\end{bmatrix}\right)^2\end{aligned}\tag{2.4}$$

for all $x,y,z\in X$ with $z\notin V(x,y)$ and $I,J\in P(N)$ with $I\cap J=\phi$.

Proof. Let $I,J\in P(N)$ with $I\cap J=\phi$. Then we have

$$\begin{aligned}\delta(I,x,y,z)&=\left(\|x,z\|^2-\sum_{k\in I\cup J}|(x,e_k|z)|^2+\sum_{j\in J}|(x,e_j|z)|^2\right)\\&\qquad\times\left(\|y,z\|^2-\sum_{k\in I\cup J}|(y,e_k|z)|^2+\sum_{j\in J}|(y,e_j|z)|^2\right)\\&\qquad-\left|(x,y|z)-\sum_{k\in I\cup J}(x,e_k|z)(e_k,y|z)+\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right|^2\\&=\left(\|x,z\|^2-\sum_{k\in I\cup J}|(x,e_k|z)|^2\right)\left(\|y,z\|^2-\sum_{k\in I\cup J}|(y,e_k|z)|^2\right)\\&\qquad+\sum_{j\in J}|(x,e_j|z)|^2\sum_{j\in J}|(y,e_j|z)|^2+\left(\|x,z\|^2-\sum_{k\in I\cup J}|(x,e_k|z)|^2\right)\sum_{j\in J}|(y,e_j|z)|^2\\&\qquad+\sum_{j\in J}|(x,e_j|z)|^2\left(\|y,z\|^2-\sum_{k\in I\cup J}|(y,e_k|z)|^2\right)\\&\qquad-\left|(x,y|z)-\sum_{k\in I\cup J}(x,e_k|z)(e_k,y|z)\right|^2-\left|\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right|^2\\&\qquad-2\left[\left((x,y|z)-\sum_{k\in I\cup J}(x,e_k|z)(e_k,y|z)\right)\sum_{j\in J}(x,e_j|z)(e_j,y|z)\right].\end{aligned}$$

Since $\{e_i\}_{i\in N}$ is a z-orthonormal family of vectors in a 2-inner product space X

$$\|w-\sum_{k\in K}(w,e_k|z)e_k\,,\,z\|^2=\|w,z\|^2-\sum_{k\in K}|(w,e_k|z)|^2,$$

and

$$\Big(w - \sum_{k\in K}(w, e_k|z)e_k, u - \sum_{k\in K}(u, e_k|z)e_k \,\Big|\, z\Big) = (w, u|z) - \sum_{k\in K} |(w, e_k|z)(e_k, u|z)$$

for all $w, u, z \in X$ with $z \notin V(x, y)$, and so we have

$$\delta(I, x, y, z) = \delta(I \cup J, x, y, z) + \gamma(J, x, y, z) + \rho(I, J, x, y, z)$$

where

$$\begin{aligned}
&\rho(I, J, x, y, z)\\
&= \|x - \sum_{k\in I\cup J}(x, e_k|z)e_k, z\|^2 \sum_{j\in J} |(y, e_j|z)|^2\\
&\quad + \sum_{j\in J} |(x, e_j|z)|^2 \|y - \sum_{k\in I\cup J}(y, e_k|z)e_k, z\|^2\\
&\quad - 2\Big[\Big(x - \sum_{k\in I\cup J}(x, e_k|z)e_k,\, y - \sum_{k\in I\cup J}(y, e_k|z)e_k \,\Big|\, z\Big) \sum_{j\in J}(x, e_j|z)(e_j, y|z)\Big]
\end{aligned}$$

and then we get

$$\begin{aligned}
&\rho(I, J, x, y, z)\\
&= \left(\det \begin{bmatrix} \|x - \sum_{k\in I\cup J}(x, e_k|z)e_k, z\| & (\sum_{j\in J} |(y, e_j|z)|^2)^{1/2} \\ \|y - \sum_{k\in I\cup J}(x, e_k|z)e_k, z\| & (\sum_{j\in J} |(y, e_j|z)|^2)^{1/2} \end{bmatrix}\right)^2\\
&\quad + 2\Big\{ \|x - \sum_{k\in I\cup J}(x, e_k|z)e_k, z\| \|y - \sum_{k\in I\cup J}(y, e_k|z)e_k, z\|\\
&\quad \times \Big(\sum_{j\in J} |(x, e_j|z)|^2\Big)^{1/2} \Big(\sum_{j\in J} |(y, e_j|z)|^2\Big)^{1/2}\\
&\quad - \Big[\Big(x - \sum_{k\in I\cup J}(x, e_k|z)e_k,\, y - \sum_{k\in I\cup J}(y, e_k|z)e_k \,\Big|z\Big) \sum_{j\in J}(x, e_j|z)(e_j, y|z)\Big]\Big\}.
\end{aligned}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. A similar argument as in Theorem 2.1 shows that the second part of $\rho(I, J, x, y, z)$ is nonnegative. Thus, we deduce the desired inequality (2.4). This completes the proof. □

Corollary 2.3. With the assumptions of Theorem 2.1, we have

$$\delta(J, x, y, z) \geq \delta(I, x, y, z) \geq 0,$$

for all $x, y, z \in X$ with $z \notin V(x, y)$ and $I, J \in P(N)$ with $I \supset J(\neq \phi)$, $I \neq J$, i.e., the mapping $\delta(\cdot, x, y, z)$ is monotonic nonincreasing on $P(N)$.

Proof. Let $I, J \in P(N)$ with $I \supset J(\neq \phi)$ and $J \neq I$. Then, by the inequality (2.4), we have the inequality

$$\delta(J, x, y, z) \geq \delta(J \cup (I \setminus J), x, y, z) + \gamma(I \setminus J, x, y, z) = \delta(I, x, y, z) + \gamma(I \setminus J, x, y, z)$$

from which we have

$$\delta(J, x, y, z) - \delta(I, x, y, z) \geq \gamma(I \setminus J, x, y, z) \geq 0$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. This completes the proof. □

Remark. By the inequality (2.4) we have

$$\delta(I, x, y, z) \geq \gamma(J, x, y, z)$$

for all $I, J \in P(N)$ with $I \cap J = \phi$.

Theorem 2.4. Let X be a 2-inner product space, $(e_i)_{1\leq i\leq n}$ be a z-orthonormal family of vectors in X and $P(N)$ be the class of all finite indices of N. Then we have

$$\begin{aligned} &\delta(I, x, y, z) + \gamma(J, x, y, z) \\ &\geq \delta(I \cup J, x, y, z) + \left(\det \begin{bmatrix} \|x - \sum_{i\in I}(x, e_i|z)e_i, z\| & (\sum_{j\in J} |(x, e_j|z)|^2)^{1/2} \\ \|y - \sum_{i\in I}(y, e_i|z)e_i, z\| & (\sum_{j\in J} |(y, e_j|z)|^2)^{1/2} \end{bmatrix}\right)^2 \end{aligned} \tag{2.5}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$ and $I, J \in P(N)$ with $I \cap J = \phi$.

Proof. Let $I, J \in P(N)$ with $I \cap J = \phi$. Then we have

$$\begin{aligned} &\delta(I \cup J, x, y, z) \\ &= \left(\|x, z\|^2 - \sum_{i\in I} |(x, e_i|z)|^2 - \sum_{j\in J} |(x, e_j|z)|^2\right) \\ &\qquad \times \left(\|y, z\|^2 - \sum_{i\in I} |(y, e_i|z)|^2 - \sum_{j\in J} |(y, e_j|z)|^2\right) \\ &\quad - \left|(x, y|z) - \sum_{i\in I}(x, e_i|z)(e_i, y|z) - \sum_{j\in J}(x, e_j|z)(e_j, y|z)\right|^2 \\ &= \delta(I, x, y, z) + \gamma(J, x, y, z) - \tau(I, J, x, y, z) \end{aligned}$$

where

$$\begin{aligned} &\tau(I, J, x, y, z) \\ &= \|x - \sum_{i\in I}(x, e_i|z)e_i, z\|^2 \sum_{j\in J} |(y, e_j|z)|^2 \\ &\quad + \sum_{j\in J} |(x, e_j|z)|^2 \|y - \sum_{i\in I}(y, e_i|z)e_i, z\|^2 \\ &\quad - 2\left[\left(x - \sum_{i\in I}(x, e_i|z)e_i\,,\ y - \sum_{i\in I}(y, e_i|z)e_i \,\Big|\, z\right) \sum_{j\in J} (x, e_j|z)(e_j, y|z)\right] \end{aligned}$$

for all $x, y, z \in X$ with $z \notin V(x, y)$. A similar argument as in Theorem 2.2 shows that

$$\tau(I, J, x, y, z) \geq \left(\det \begin{bmatrix} \|x - \sum_{i\in I}(x, e_i|z)e_i, z\| & (\sum_{j\in J} |(y, e_j|z)|^2)^{1/2} \\ \|y - \sum_{i\in I}(x, e_i|z)e_i, z\| & (\sum_{j\in J} |(y, e_j|z)|^2)^{1/2} \end{bmatrix}\right)^2,$$

which proves the inequality (2.5). This completes the proof. □

Corollary 2.5. With the assumptions of Theorem 2.1, we have

$$\delta(I \cup J, x, y, z) \leq \delta(I, x, y, z) + \delta(J, x, y, z) \tag{2.6}$$

for all $x, y, z \in X$ and $z \notin V(x, y)$ and $I, J \in P(N)$ with $I \cap J = \phi$, i.e., the mapping $\delta(\cdot, x, y, z)$ is subadditive on $P(N)$.

Proof. Let $I, J \in P(N)$ with $I \cap J = \phi$. Then, by the inequality (2.5), we have

$$\delta(I \cup J, x, y, z) \leq \delta(I, x, y, z) + \gamma(J, x, y, z)$$

and

$$\delta(J \cup I), x, y, z) \leq \delta(J, x, y, z) + \gamma(I, x, y, z),$$

which gives by addition

$$2\delta(I \cup J, x, y, z) \leq \delta(I, x, y, z) + \delta(J, x, y, z) + \gamma(I, x, y, z) + \gamma(J, x, y, z). \tag{2.7}$$

By Remark, we have

$$\delta(I, x, y, z) \geq \gamma(J, x, y, z) \qquad \text{and} \qquad \delta(J, x, y, z) \geq \gamma(I, x, y, z)$$

and so

$$\delta(I, x, y, z) + \delta(J, x, y, z) \geq \gamma(I, x, y, z) + \gamma(J, x, y, z).$$

Thus, by the inequality (2.7) we deduce

$$2\delta(I \cup J, x, y, z) \leq 2[\delta(I, x, y, z) + \delta(J, x, y, z)].$$

This completes the proof. □

Theorem 2.6. Let X be a 2-inner product space, $(e_i)_{1 \leq i \leq n}$ be a z-orthonormal family of vectors in X and $P(N)$ be the class of all finite indices of N. Then we have

$$\delta(I, x, y, z) + \left(\det \begin{bmatrix} \|x, z\| & (\sum_{i \in I} |(x, e_i|z)|)^{1/2} \\ \|y, z\| & (\sum_{i \in I} |(y, e_i|z)|)^{1/2} \end{bmatrix} \right)^2 \leq \Gamma_z(x, y) + \gamma_z(I, x, y, z) \tag{2.8}$$

where $\Gamma_z(x, y)$ is the Gram determinant of x, y with respect to z ([7]), i.e.,

$$\Gamma_z(x, y) = \det \begin{bmatrix} (x, x|z) & (x, y|z) \\ (y, x|z) & (y, y|z) \end{bmatrix} = \|x, z\|^2 \|y, z\|^2 - |(x, y|z)|^2$$

for all $x, y, z \in X$ with $z \notin V(x, y)$.

Proof. We easily obtain the following

$$\begin{aligned} &\delta(I, x, y, z) \\ &= \|x, z\|^2 \|y, z\|^2 - \|x, z\|^2 \sum_{i \in I} |(y, e_i|z)|^2 - \|y, z\|^2 \sum_{i \in I} |(x, e_i|z)|^2 \\ &\quad + \sum_{i \in I} |(x, e_i|z)|^2 \sum_{i \in I} |(y, e_i|z)|^2 - |(x, y|z)|^2 \\ &\quad + 2\left[(x, y|z) \sum_{i \in I} (x, e_i|z)(e_i, y|z) \right] - \left| \sum_{i \in I} (x, e_i|z)(e_i, y|z) \right|^2 \\ &= \Gamma_z(x, y) + \gamma(I, x, y, z) - t(I, x, y, z) \end{aligned}$$

where

$$t(I,x,y,z) = \|y,z\|^2 \sum_{i\in I} |(x,e_i|z)|^2 + \|x,z\|^2 \sum_{i\in I} |(y,e_i|z)|^2 - 2\left[(x,y|z)\sum_{i\in I}(x,e_i|z)(e_i,y|z)\right]$$

for all $x,y,z \in X$ with $z \notin V(x,y)$. A similar argument as in Theorem 2.4 shows that

$$t(I,x,y,z) \geq \left(\det\begin{bmatrix} \|x,z\| & (\sum_{i\in I}|(y,e_i|z)|)^{1/2} \\ \|y,z\| & (\sum_{i\in I}|(y,e_i|z)|)^{1/2}\end{bmatrix}\right)^2$$

which gives the desired inequality (2.8). This completes the proof. □

Corollary 2.8. With the assumptions of Theorem 2.1, we have

$$\begin{aligned} &\|x,z\|^2\|y,z\|^2 - |(x,y|z)|^2 \\ &\geq \frac{1}{2}\left[\left(\det\begin{bmatrix} \|x,z\| & (\sum_{i\in I}|(y,e_i|z)|)^{1/2} \\ \|y,z\| & (\sum_{i\in I}|(y,e_i|z)|)^{1/2}\end{bmatrix}\right)^2\right. \\ &\qquad \left.+\left(\det\begin{bmatrix} \|x,z\| & (\sum_{j\in J}|(y,e_j|z)|)^{1/2} \\ \|y,z\| & (\sum_{j\in J}|(y,e_j|z)|2)^{1/2}\end{bmatrix}\right)^2\right] \geq 0 \end{aligned} \tag{2.9}$$

for all $x,y,z \in X$ with $z \notin V(x,y)$ and $I,J \in P(N)$ with $I \cap J = \phi$.

Proof. By the inequality (2.8), we have

$$\begin{aligned} &\delta(I,x,y,z) + \delta(J,x,y,z) \\ &\quad + \left(\det\begin{bmatrix} \|x,z\| & (\sum_{i\in I}|(x,e_i|z)|)^{1/2} \\ \|y,z\| & (\sum_{i\in I}|(y,e_i|z)|)^{1/2}\end{bmatrix}\right)^2 + \left(\det\begin{bmatrix} \|x,z\| & (\sum_{j\in J}|(x,e_j|z)|)^{1/2} \\ \|y,z\| & (\sum_{j\in J}|(y,e_j|z)|)^{1/2}\end{bmatrix}\right)^2 \\ &\leq 2\Gamma_z(x,y) + \gamma(I,x,y,z) + \gamma(J,x,y,z). \end{aligned}$$

By Remark, we have

$$\gamma(I,x,y,z) + \gamma(J,x,y,z) \leq \delta(I,x,y,z) + \delta(J,x,y,z)$$

for all $I,J \in P(N)$ with $I \cap J = \phi$. This completes the proof. □

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