SOME JENSEN'S TYPE INEQUALITIES FOR LOG-CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [5, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [5] and the references therein. For other results, see [11], [6], [10] and [8]. For recent results, see [2] and [3].

Date: January 03, 2010.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Positive operators, Jensen's inequality, Convex functions, Functions of selfadjoint operators.

2. Some Jensen's Type Inequalities for Log-convex Functions

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [9] (see also [5, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [9])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f : I \to (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1-t)a + tb) \leq [f(a)]^{1-t}[f(b)]^{t}$$

for any $a, b \in I$ and $t \in [0, 1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

As an imediate consequence of the Mond-Pečarić inequality above we can provide the following result:

Theorem 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$g(\langle Ax, x\rangle) \leq \exp\langle \ln g(A)x, x\rangle \leq \langle g(A)x, x\rangle \tag{2.1}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Consider the function $f := \ln g$, which is convex on $[m, M]$. Writing (MP) for f we get $\ln[g(\langle Ax, x\rangle)] \leq \langle \ln g(A)x, x\rangle$, for each $x \in H$ with $\|x\| = 1$, which, by taking the exponential, produces the first inequality in (2.1).

If we also use (MP) for the exponential function, we get

$$\exp\langle \ln g(A)x, x\rangle \leq \langle \exp[\ln g(A)]x, x\rangle = \langle g(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$ and the proof is complete. □

The case of sequences of operators may be of interest and is embodied in the following corollary:

Corollary 1. *Assume that g is as in the Theorem 2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then*

$$g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \leq \exp\left\langle \sum_{j=1}^{n} \ln g(A_j)x_j, x_j\right\rangle \leq \left\langle \sum_{j=1}^{n} g(A_j)x_j, x_j\right\rangle. \tag{2.2}$$

Proof. As in [5, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$,

$$\left\langle g\left(\widetilde{A}\right)\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle g(A_j)\, x_j, x_j\rangle, \left\langle \widetilde{A}\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle$$

and so on.

Applying Theorem 2 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result (2.2). □

In particular we have:

Corollary 2. *Assume that g is as in the Theorem 2. If A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) \leq \left\langle \prod_{j=1}^{n} [g(A_j)]^{p_j} x, x\right\rangle \leq \left\langle \sum_{j=1}^{n} p_j g(A_j)\, x, x\right\rangle \tag{2.3}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows from Corollary 1 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. □

It is also important to observe that, as a special case of Theorem 1 we have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

Theorem 3 (Hölder-McCarthy, 1967, [7])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all $r > 1$ and $x \in H$ with $\|x\| = 1$;*

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;*

(iii) If A is invertible, then $\langle A^{-r} x, x\rangle \geq \langle Ax, x\rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, we can improve the Hölder-McCarthy inequality as follows:

Proposition 1. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$\langle Ax, x\rangle^{-r} \leq \exp\left\langle \ln\left(A^{-r}\right) x, x\right\rangle \leq \left\langle A^{-r} x, x\right\rangle \tag{2.4}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [5, p. 57]:

Theorem 4. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A)x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot f(M) \tag{2.5}$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows:

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$\begin{aligned}\langle g(A)x, x\rangle &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}} \right] x, x \right\rangle \\ &\leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot g(M)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}g(\langle Ax, x\rangle) &\leq [g(m)]^{\frac{M - \langle Ax, x\rangle}{M-m}} [g(M)]^{\frac{\langle Ax, x\rangle - m}{M-m}} \\ &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}} \right] x, x \right\rangle\end{aligned} \tag{2.7}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Observe that, by the log-convexity of g, we have

$$g(t) = g\left(\frac{M - t}{M - m} \cdot m + \frac{t - m}{M - m} \cdot M \right) \leq [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \tag{2.8}$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A, we have that

$$\langle g(A)x, x\rangle \leq \langle \Psi(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$, where $\Psi(t) := [g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}}$, $t \in [m, M]$. This proves the first inequality in (2.6).

Now, observe that, by the weighted arithmetic mean-geometric mean inequality we have

$$[g(m)]^{\frac{M-t}{M-m}} [g(M)]^{\frac{t-m}{M-m}} \leq \frac{M - t}{M - m} \cdot g(m) + \frac{t - m}{M - m} \cdot g(M)$$

for any $t \in [m, M]$.

Applying the property (P) for the operator A we deduce the second inequality in (2.6).

Further on, if we use the inequality (2.8) for $t = \langle Ax, x\rangle \in [m, M]$ then we deduce the first part of (2.7).

Now, observe that the function Ψ introduced above can be rearranged to read as

$$\Psi(t) = g(m) \left[\frac{g(M)}{g(m)} \right]^{\frac{t-m}{M-m}}, t \in [m, M]$$

showing that Ψ is a convex function on $[m, M]$.

Applying Mond-Pečarić's inequality for Ψ we deduce the second part of (2.7) and the proof is complete. □

The case of sequences of operators is as follows:

Corollary 3. *Assume that* g *is as in the Theorem 2. If* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ *and* $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then*

$$\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \leq \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} \right] x_j, x_j \right\rangle \tag{2.9}$$
$$\leq \frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M-m} \cdot g(m) + \frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M-m} \cdot g(M)$$

and

$$g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \leq [g(m)]^{\frac{M - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle}{M-m}} [g(M)]^{\frac{\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - m}{M-m}} \tag{2.10}$$
$$\leq \left\langle \sum_{j=1}^{n} \left[[g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} \right] x_j, x_j \right\rangle.$$

In particular we have:

Corollary 4. *Assume that* g *is as in the Theorem 2. If* A_j *are selfadjoint operators with* $Sp(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} x, x \right\rangle \tag{2.11}$$
$$\leq \frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M-m} \cdot g(m) + \frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M-m} \cdot g(M)$$

and

$$g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) \leq [g(m)]^{\frac{M - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}{M-m}} [g(M)]^{\frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle - m}{M-m}} \tag{2.12}$$
$$\leq \left\langle \sum_{j=1}^{n} p_j [g(m)]^{\frac{M1_H - A_j}{M-m}} [g(M)]^{\frac{A_j - m1_H}{M-m}} x, x \right\rangle.$$

The above result from Theorem 5 can be utilized to produce the following reverse inequality for negative powers of operators:

Proposition 2. *Let* A *be a selfadjoint positive operator on a Hilbert space* H. *If* A *is invertible and* $Sp(A) \subseteq [m, M]\ (0 < m < M)$, *then*

$$\langle A^{-r} x, x \rangle \leq \left\langle \left[m^{\frac{M1_H - A}{M-m}} M^{\frac{A - m1_H}{M-m}} \right]^{-r} x, x \right\rangle \tag{2.13}$$
$$\leq \frac{M - \langle Ax, x \rangle}{M-m} \cdot m^{-r} + \frac{\langle Ax, x \rangle - m}{M-m} \cdot M^{-r}$$

and

$$\langle Ax,x\rangle^{-r}\leq\left[g(m)^{\frac{M-\langle Ax,x\rangle}{M-m}}g(M)^{\frac{\langle Ax,x\rangle-m}{M-m}}\right]^{-r}\leq\left\langle\left[m^{\frac{M1_H-A}{M-m}}M^{\frac{A-m1_H}{M-m}}\right]^{-r}x,x\right\rangle \tag{2.14}$$

for all $r>0$ and $x\in H$ with $\|x\|=1$.

3. Jensen's Inequality for Differentiable Log-convex Functions

The following result provides a reverse for the Jensen type inequality (MP):

Theorem 6 (Dragomir, 2008, [4]). *Let J be an interval and $f:J\to\mathbb{R}$ be a convex and differentiable function on $\mathring{J}$ (the interior of J) whose derivative f' is continuous on $\mathring{J}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A)\subseteq[m,M]\subset\mathring{J}$, then*

$$(0\leq)\langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\leq\langle f'(A)Ax,x\rangle-\langle Ax,x\rangle\cdot\langle f'(A)x,x\rangle \tag{3.1}$$

for any $x\in H$ with $\|x\|=1$.

The following result may be stated:

Proposition 3. *Let J be an interval and $g:J\to\mathbb{R}$ be a differentiable log-convex function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A)\subseteq[m,M]\subset\mathring{J}$, then*

$$(1\leq)\frac{\exp\langle\ln g(A)x,x\rangle}{g(\langle Ax,x\rangle)}\leq\exp\left[\left\langle g'(A)[g(A)]^{-1}Ax,x\right\rangle-\langle Ax,x\rangle\cdot\left\langle g'(A)[g(A)]^{-1}x,x\right\rangle\right] \tag{3.2}$$

for each $x\in H$ with $\|x\|=1$.

Proof. It follows by the inequality (3.1) written for the convex function $f=\ln g$ that

$$\langle\ln g(A)x,x\rangle\leq\ln g(\langle Ax,x\rangle)+\left\langle g'(A)[g(A)]^{-1}Ax,x\right\rangle-\langle Ax,x\rangle\cdot\left\langle g'(A)[g(A)]^{-1}x,x\right\rangle$$

for each $x\in H$ with $\|x\|=1$.

Now, taking the exponential and dividing by $g(\langle Ax,x\rangle)>0$ for each $x\in H$ with $\|x\|=1$, we deduce the desired result (3.2). □

Corollary 5. *Assume that g is as in the Proposition 3 and A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]\subset\mathring{J}$, $j\in\{1,...,n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$, *then*

$$(1 \leq) \frac{\exp \left\langle \sum_{j=1}^n \ln g(A_j) x_j, x_j \right\rangle}{g\left(\sum_{j=1}^n \langle A_j x, x_j \rangle\right)} \leq \exp\left[\left\langle \sum_{j=1}^n g'(A_j)[g(A_j)]^{-1} A_j x_j, x_j \right\rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \cdot \sum_{j=1}^n \left\langle g'(A_j)[g(A_j)]^{-1} x_j, x_j \right\rangle \right]. \tag{3.3}$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n p_j = 1$, *then*

$$(1 \leq) \frac{\left\langle \prod_{j=1}^n [g(A_j)]^{p_j} x, x \right\rangle}{g\left(\left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle\right)} \leq \exp\left[\left\langle \sum_{j=1}^n p_j g'(A_j)[g(A_j)]^{-1} A_j x, x \right\rangle - \sum_{j=1}^n p_j \langle A_j x, x \rangle \cdot \sum_{j=1}^n p_j \left\langle g'(A_j)[g(A_j)]^{-1} x, x \right\rangle \right] \tag{3.4}$$

for each $x \in H$ *with* $\|x\| = 1$.

Remark 1. *Let* A *be a selfadjoint positive operator on a Hilbert space* H. *If* A *is invertible, then*

$$(1 \leq) \langle Ax, x \rangle^r \exp \left\langle \ln\left(A^{-r}\right) x, x \right\rangle \leq \exp\left[r\left(\langle Ax, x \rangle \cdot \left\langle A^{-1} x, x \right\rangle - 1\right)\right] \tag{3.5}$$

for all $r > 0$ *and* $x \in H$ *with* $\|x\| = 1$.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality can be stated as well:

Theorem 7. *Let* J *be an interval and* $g : J \to \mathbb{R}$ *be a log-convex differentiable function on* $\mathring{J}$ *whose derivative* g' *is continuous on* $\mathring{J}$. *If* A *is a selfadjoint operators on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subset \mathring{J}$, *then*

$$1 \leq \left\langle \exp\left[\frac{g'(\langle Ax, x \rangle)}{g(\langle Ax, x \rangle)}(A - \langle Ax, x \rangle 1_H)\right] x, x \right\rangle \leq \frac{\langle g(A) x, x \rangle}{g(\langle Ax, x \rangle)} \leq \left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x \rangle 1_H)\right] x, x \right\rangle \tag{3.6}$$

for each $x \in H$ *with* $\|x\| = 1$, *where* 1_H *denotes the identity operator on* H.

Proof. It is well known that if $h : J \to \mathbb{R}$ is a convex differentiable function on $\mathring{J}$, then the following *gradient inequality* holds

$$h(t) - h(s) \geq h'(s)(t - s)$$

for any $t, s \in \mathring{J}$.

Now, if we write this inequality for the convex function $h = \ln g$, then we get

$$\ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t-s) \tag{3.7}$$

which is equivalent with

$$g(t) \geq g(s) \exp\left[\frac{g'(s)}{g(s)}(t-s)\right] \tag{3.8}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (3.8), then we get

$$g(t) \geq g(\langle Ax, x\rangle) \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A and the Mond-Pečarić inequality for the exponential function, we can state the following inequality that is of interest in itself as well:

$$\begin{aligned} \langle g(A)y, y\rangle &\geq g(\langle Ax, x\rangle)\left\langle \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(A - \langle Ax, x\rangle 1_H)\right]y, y\right\rangle \\ &\geq g(\langle Ax, x\rangle)\exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(\langle Ay, y\rangle - \langle Ax, x\rangle)\right] \end{aligned} \tag{3.9}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Further, if we put $y = x$ in (3.9), then we deduce the first and the second inequality in (3.6).

Now, if we replace s with t in (3.8) we can also write the inequality

$$g(t) \exp\left[\frac{g'(t)}{g(t)}(s-t)\right] \leq g(s)$$

which is equivalent with

$$g(t) \leq g(s) \exp\left[\frac{g'(t)}{g(t)}(t-s)\right] \tag{3.10}$$

for any $t, s \in \mathring{J}$.

Further, if we take $s := \langle Ax, x\rangle \in [m, M] \subset \mathring{J}$, for a fixed $x \in H$ with $\|x\| = 1$, in the inequality (3.10), then we get

$$g(t) \leq g(\langle Ax, x\rangle) \exp\left[\frac{g'(t)}{g(t)}(t - \langle Ax, x\rangle)\right]$$

for any $t \in \mathring{J}$.

Utilising the property (P) for the operator A, then we can state the following inequality that is of interest in itself as well:

$$\langle g(A)y, y\rangle \leq g(\langle Ax, x\rangle)\left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right]y, y\right\rangle \tag{3.11}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

Finally, if we put $y = x$ in (3.11), then we deduce the last inequality in (3.6). □

The case of operator sequences is embodied in the following corollary:

Corollary 6. *Assume that g is as in the Proposition 3 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$
\begin{aligned}
(3.12)\quad 1 &\leq \left\langle \sum_{j=1}^{n} \exp\left[\frac{g'\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)}{g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)} \left(A_j - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle 1_H \right) \right] x_j, x_j \right\rangle \\
&\leq \frac{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle}{g\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)} \\
&\leq \left\langle \sum_{j=1}^{n} \exp\left[g'(A_j) [g(A_j)]^{-1} \left(A_j - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle 1_H \right) \right] x_j, x_j \right\rangle.
\end{aligned}
$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$
\begin{aligned}
(3.13)\quad 1 &\leq \left\langle \sum_{j=1}^{n} p_j \exp\left[\frac{g'\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)}{g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)} \right.\right. \\
&\qquad \left.\left. \times \left(A_j - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle 1_H \right) \right] x, x \right\rangle \\
&\leq \frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle}{g\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right)} \\
&\leq \left\langle \sum_{j=1}^{n} p_j \exp\left[g'(A_j) [g(A_j)]^{-1} \left(A_j - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle 1_H \right) \right] x, x \right\rangle.
\end{aligned}
$$

Remark 2. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then*

$$
\begin{aligned}
(3.14)\quad 1 &\leq \left\langle \exp\left[r\left(1_H - \langle Ax, x \rangle^{-1} A\right) \right] x, x \right\rangle \\
&\leq \left\langle A^{-r} x, x \right\rangle \langle Ax, x \rangle^{r} \leq \left\langle \exp\left[r\left(1_H - \langle Ax, x \rangle A^{-1}\right) \right] x, x \right\rangle
\end{aligned}
$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

The following reverse inequality may be proven as well:

Theorem 8. *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operators* *on the Hilbert space H with $Sp(A) \subseteq [m,M] \subset \mathring{J}$, then*

$$(1 \leq) \frac{\left\langle [g(M)]^{\frac{A-m1_H}{M-m}} [g(m)]^{\frac{M1_H-A}{M-m}} x, x \right\rangle}{\langle g(A)x,x\rangle} \leq \frac{\left\langle g(A) \exp\left[\frac{(M1_H-A)(A-m1_H)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] x, x\right\rangle}{\langle g(A)x,x\rangle} \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \tag{3.15}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Utilising the inequality (3.7) we have successively

$$\frac{g((1-\lambda)t + \lambda s)}{g(s)} \geq \exp\left[(1-\lambda)\frac{g'(s)}{g(s)}(t-s)\right] \tag{3.16}$$

and

$$\frac{g((1-\lambda)t + \lambda s)}{g(t)} \geq \exp\left[-\lambda\frac{g'(t)}{g(t)}(t-s)\right] \tag{3.17}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Now, if we take the power λ in the inequality (3.16) and the power $1-\lambda$ in (3.17) and multiply the obtained inequalities, we deduce

$$\frac{[g(t)]^{1-\lambda}[g(s)]^{\lambda}}{g((1-\lambda)t+\lambda s)} \leq \exp\left[(1-\lambda)\lambda\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t-s)\right] \tag{3.18}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0,1]$.

Further on, if we choose in (3.18) $t = M, s = m$ and $\lambda = \frac{M-u}{M-m}$, then, from (3.18) we get the inequality

$$\frac{[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}}}{g(u)} \leq \exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \tag{3.19}$$

which, together with the inequality

$$\frac{(M-u)(u-m)}{M-m} \leq \frac{1}{4}(M-m)$$

produce

$$[g(M)]^{\frac{u-m}{M-m}}[g(m)]^{\frac{M-u}{M-m}} \leq g(u)\exp\left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \leq g(u)\exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \tag{3.20}$$

for any $u \in [m,M]$.

If we apply the property (P) to the inequality (3.20) and for the operator A we deduce the desired result. □

Corollary 7. *Assume that g is as in the Theorem 8 and A_j are selfadjoint operators with $Sp(A_j) \subseteq [m, M] \subset \mathring{J}$, $j \in \{1, ..., n\}$.*

If $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, then

$$(1 \leq) \frac{\sum_{j=1}^{n} \left\langle [g(M)]^{\frac{A_j - m1_H}{M-m}} [g(m)]^{\frac{M1_H - A_j}{M-m}} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \tag{3.21}$$
$$\leq \frac{\sum_{j=1}^{n} \left\langle g(A_j) \exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] x_j, x_j \right\rangle}{\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle}$$
$$\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right].$$

If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then for each $x \in H$ with $\|x\| = 1$

$$(1 \leq) \frac{\left\langle \sum_{j=1}^{n} p_j [g(M)]^{\frac{A_j - m1_H}{M-m}} [g(m)]^{\frac{M1_H - A_j}{M-m}} x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle} \tag{3.22}$$
$$\leq \frac{\left\langle \sum_{j=1}^{n} p_j g(A_j) \exp\left[\frac{(M1_H - A_j)(A_j - m1_H)}{M-m} \left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] x, x \right\rangle}{\left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle}$$
$$\leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right].$$

Remark 3. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp(A) \subseteq [m, M]$ $(0 < m < M)$, then*

$$(1 \leq) \frac{\left\langle [g(M)]^{\frac{r(m1_H - A)}{M-m}} [g(m)]^{\frac{r(A - M1_H)}{M-m}} x, x \right\rangle}{\langle A^{-r} x, x \rangle} \tag{3.23}$$
$$\leq \frac{\left\langle A^{-r} \exp\left[\frac{r(M1_H - A)(A - m1_H)}{Mm}\right] x, x \right\rangle}{\langle A^{-r} x, x \rangle} \leq \exp\left[\frac{1}{4} r \frac{(M-m)^2}{mM}\right]$$

4. Applications for Ky Fan's Inequality

Consider the function $g : (0,1) \to \mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r$, $r > 0$. Observe that for the new function $f : (0,1) \to \mathbb{R}$, $f(t) = \ln g(t)$ we have

$$f'(t) = \frac{-r}{t(1-t)} \text{ and } f''(t) = \frac{2r\left(\frac{1}{2} - t\right)}{t^2 (1-t)^2} \text{ for } t \in (0,1)$$

showing that the function g is log-convex on the interval $\left(0, \frac{1}{2}\right)$.

If $p_i > 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $t_i \in \left(0, \frac{1}{2}\right)$ for $i \in \{1, ..., n\}$, then by applying the Jensen inequality for the convex function f (with $r = 1$) on the interval $\left(0,\frac{1}{2}\right)$ we get

$$\frac{\sum_{i=1}^{n}p_it_i}{1-\sum_{i=1}^{n}p_it_i}\geq\prod_{i=1}^{n}\left(\frac{t_i}{1-t_i}\right)^{p_i}, \tag{4.1}$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [1, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^{n}\left(\frac{1-t_i}{t_i}\right)^{p_i}\geq\frac{1-\sum_{i=1}^{n}p_it_i}{\sum_{i=1}^{n}p_it_i},$$

where $p_i>0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^{n}p_i=1$ and $t_i\in\left(0,\frac{1}{2}\right)$ for $i\in\{1,...,n\}$.

By the weighted arithmetic mean - geometric mean inequality we also have that

$$\sum_{i=1}^{n}p_i\left(1-t_i\right)t_i^{-1}\geq\prod_{i=1}^{n}\left(\frac{1-t_i}{t_i}\right)^{p_i}$$

giving the double inequality

$$\sum_{i=1}^{n}p_i\left(1-t_i\right)t_i^{-1}\geq\prod_{i=1}^{n}\left(\left(1-t_i\right)t_i^{-1}\right)^{p_i}\geq\sum_{i=1}^{n}p_i\left(1-t_i\right)\left(\sum_{i=1}^{n}p_it_i\right)^{-1}. \tag{4.2}$$

The following operator inequalities generalizing (4.2) may be stated:

Proposition 4. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp\left(A\right)\subset\left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}\left\langle\left(A^{-1}\left(1_H-A\right)\right)^r x,x\right\rangle&\geq\exp\left\langle\ln\left(A^{-1}\left(1_H-A\right)\right)^r x,x\right\rangle\\&\geq\left(\left\langle\left(1_H-A\right)x,x\right\rangle\left\langle Ax,x\right\rangle^{-1}\right)^r\end{aligned} \tag{4.3}$$

for each $x\in H$ with $\|x\|=1$ and $r>0$.

In particular,

$$\begin{aligned}\left\langle A^{-1}\left(1_H-A\right)x,x\right\rangle&\geq\exp\left\langle\ln\left(A^{-1}\left(1_H-A\right)\right)x,x\right\rangle\\&\geq\left\langle\left(1_H-A\right)x,x\right\rangle\left\langle Ax,x\right\rangle^{-1}\end{aligned} \tag{4.4}$$

for each $x\in H$ with $\|x\|=1$.

The proof follows by Theorem 2 applied for the log-convex function $g\left(t\right)=\left(\frac{1-t}{t}\right)^r, r>0, t\in\left(0,\frac{1}{2}\right)$.

Proposition 5. *Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible and $Sp\left(A\right)\subseteq\left[m,M\right]\subset\left(0,\frac{1}{2}\right)$, then*

$$\begin{aligned}&\left\langle\left(\left(1_H-A\right)A^{-1}\right)^r x,x\right\rangle\\&\leq\left\langle\left[\left(\frac{1-m}{m}\right)^{\frac{r\left(M1_H-A\right)}{M-m}}\left(\frac{1-M}{M}\right)^{\frac{r\left(A-m1_H\right)}{M-m}}\right]x,x\right\rangle\\&\leq\frac{M-\left\langle Ax,x\right\rangle}{M-m}\cdot\left(\frac{1-m}{m}\right)^r+\frac{\left\langle Ax,x\right\rangle-m}{M-m}\cdot\left(\frac{1-M}{M}\right)^r\end{aligned} \tag{4.5}$$

and

$$\left(\frac{1-\langle Ax,x\rangle}{\langle Ax,x\rangle}\right)^{r}\leq\left(\frac{1-m}{m}\right)^{\frac{r(M-\langle Ax,x\rangle)}{M-m}}\left(\frac{1-M}{M}\right)^{\frac{r(\langle Ax,x\rangle-m)}{M-m}}\leq\left\langle\left[\left(\frac{1-m}{m}\right)^{\frac{r(M1_H-A)}{M-m}}\left(\frac{1-M}{M}\right)^{\frac{r(A-m1_H)}{M-m}}\right]x,x\right\rangle \tag{4.6}$$

for each $x\in H$ *with* $\|x\|=1$ *and* $r>0$.

The proof follows by Theorem 5 applied for the log-convex function $g(t)=\left(\frac{1-t}{t}\right)^{r}, r>0, t\in\left(0,\frac{1}{2}\right)$.

Finally we have:

Proposition 6. *Let* A *be a selfadjoint positive operator on a Hilbert space* H. *If* A *is invertible and* $Sp(A)\subset\left(0,\frac{1}{2}\right)$, *then*

$$(1\leq)\,\frac{\exp\left\langle\ln\left(\left(1_H-A\right)A^{-1}\right)^{r}x,x\right\rangle}{\left(\left(1-\langle Ax,x\rangle\right)\langle Ax,x\rangle^{-1}\right)^{r}}\leq\exp\left[r\left(\langle Ax,x\rangle\cdot\left\langle A^{-1}\left(1_H-A\right)^{-1}x,x\right\rangle-\left\langle\left(1_H-A\right)^{-1}x,x\right\rangle\right)\right] \tag{4.7}$$

and

$$1\leq\left\langle\exp\left[r\left(1-\langle Ax,x\rangle\right)^{-1}\left(1_H-\langle Ax,x\rangle^{-1}A\right)\right]x,x\right\rangle\leq\frac{\left\langle\left(\left(1_H-A\right)A^{-1}\right)^{r}x,x\right\rangle}{\left(\left(1-\langle Ax,x\rangle\right)\langle Ax,x\rangle^{-1}\right)^{r}}\leq\left\langle\exp\left[r\left(1_H-A\right)^{-1}\left(\langle Ax,x\rangle A^{-1}-1_H\right)\right]x,x\right\rangle \tag{4.8}$$

for each $x\in H$ *with* $\|x\|=1$ *and* $r>0$.

The proof follows by Proposition 3 and Theorem 7 applied for the log-convex function $g(t)=\left(\frac{1-t}{t}\right)^{r}, r>0, t\in\left(0,\frac{1}{2}\right)$. The details are omitted.

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Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://www.staff.vu.edu.au/rgmia/dragomir/