# SOME BOUNDS FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. On utilising the spectral representation of selfadjoint operators in Hilbert spaces, some inequalities for continuous functions of such operators are given.

## 1. Introduction

Let $A$ be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism $\Phi$ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of $A$, denoted $Sp(A)$, and the $C^*$-algebra $C^*(A)$ generated by $A$ and the identity operator $1_H$ on $H$ as follows (see for instance [4, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator $A$.

If $A$ is a selfadjoint operator and $f$ is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a *positive operator* on $H$. Moreover, if both $f$ and $g$ are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [4] and the references therein.

For other recent results see [1], [2], [3], [6], [8], [9] and [10].

Let $U$ be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $Sp(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{R}$, it is well known that we have the following *spectral representation in terms of the*

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*Riemann-Stieltjes integral*:

$$\langle f(U)x,y\rangle=\int_{m-0}^{M}f(\lambda)\,d(\langle E_\lambda x,y\rangle), \tag{1.1}$$

for any $x,y\in H$. The function $g_{x,y}(\lambda):=\langle E_\lambda x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,y\in H$. It is also well known that $g_x(\lambda):=\langle E_\lambda x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some inequalities for continuous functions of such operators are given.

## 2. Some Vector Inequalities

The following result holds:

**Theorem 1.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\begin{aligned}|f(M)\langle x,y\rangle-\langle f(A)x,y\rangle| &\le \max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right]\bigvee_m^M(f)\\ &\le \|x\|\,\|y\|\bigvee_m^M(f)\end{aligned} \tag{2.1}$$

*for any $x,y\in H$.*

*Proof.* Integrating by parts in the Riemann-Stieltjes integral, we have that

$$\begin{aligned}\int_{m-0}^{M}f(\lambda)\,d(\langle E_\lambda x,y\rangle) &= f(\lambda)\langle E_\lambda x,y\rangle\big|_{m-0}^{M}-\int_{m-0}^{M}\langle E_\lambda x,y\rangle\,d(f(\lambda))\\ &= f(M)\langle x,y\rangle-\int_{m-0}^{M}\langle E_\lambda x,y\rangle\,d(f(\lambda))\end{aligned}$$

which, together with (1.1), produces the equality

$$f(M)\langle x,y\rangle-\langle f(A)x,y\rangle=\int_{m-0}^{M}\langle E_\lambda x,y\rangle\,d(f(\lambda)). \tag{2.2}$$

It is well that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of $v$ on $[a,b]$.

Now, if we use this property, then by (2.2) we get the following inequality that is of interest in itself:

$$|f(M)\langle x,y\rangle - \langle f(A)x,y\rangle| = \left|\int_{m-0}^{M} \langle E_\lambda x,y\rangle \, d(f(\lambda))\right| \leq \max_{\lambda\in[m,M]} |\langle E_\lambda x,y\rangle| \bigvee_{m}^{M}(f). \tag{2.3}$$

If $P$ is a nonnegative operator on $H$, i.e., $\langle Px,x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in $H$

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle \langle Py,y\rangle \tag{2.4}$$

for any $x,y \in H$.

Further on, by utilizing this inequality we have

$$\max_{\lambda\in[m,M]} |\langle E_\lambda x,y\rangle| \leq \max_{\lambda\in[m,M]} \left[\langle E_\lambda x,x\rangle^{1/2} \langle E_\lambda y,y\rangle^{1/2}\right] \leq \max_{\lambda\in[m,M]} \langle E_\lambda x,x\rangle^{1/2} \max_{\lambda\in[m,M]} \langle E_\lambda y,y\rangle^{1/2} = \|x\| \|y\|$$

which together with (2.3) produces the desired result (2.1). □

The following result also holds

**Theorem 2.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m,M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m,M]$, then we have the inequality*

$$|f(M)\langle x,y\rangle - \langle f(A)x,y\rangle| \leq L\int_{m-0}^{M} \langle E_t x,x\rangle^{1/2} \langle E_t y,y\rangle^{1/2} dt \leq L\left(M\|x\|^2 - \langle Ax,x\rangle\right)^{1/2}\left(M\|y\|^2 - \langle Ay,y\rangle\right)^{1/2} \tag{2.5}$$

*for any $x,y \in H$.*

*Proof.* It is well that if $p : [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s-t| \text{ for any } t,s \in [a,b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have from (2.2)

$$|f(M)\langle x,y\rangle - \langle f(A)x,y\rangle| = \left|\int_{m-0}^{M} \langle E_t x,y\rangle\, d(f(t))\right| \leq L\int_{m-0}^{M} |\langle E_t x,y\rangle|\, dt \tag{2.6}$$

for any $x,y \in H$.

By utilizing the generalized Schwarz inequality for nonnegative operators (2.4) and the Cauchy-Buniakovski-Schwarz inequality for the Riemann integral we have

$$\int_{m-0}^{M} |\langle E_t x, y\rangle| \, dt \le \int_{m-0}^{M} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \, dt \le \left(\int_{m-0}^{M} \langle E_t x, x\rangle \, dt\right)^{1/2} \left(\int_{m-0}^{M} \langle E_t y, y\rangle \, dt\right)^{1/2} \tag{2.7}$$

for any $x, y \in H$.

On the other hand, integrating by parts we also have

$$\int_{m-0}^{M} \langle E_t x, x\rangle \, dt = t \langle E_t x, x\rangle|_{m-0}^{M} - \int_{m-0}^{M} t d(\langle E_t x, x\rangle) = M\|x\|^2 - \langle Ax, x\rangle \tag{2.8}$$

and

$$\int_{m-0}^{M} \langle E_t y, y\rangle \, dt = M\|y\|^2 - \langle Ay, y\rangle \tag{2.9}$$

for any $x, y \in H$.

On making use of (2.6)-(2.9) we deduce the desired result (2.5). □

The following lemma may be stated.

**Lemma 1.** *Let $u : [a, b] \to \mathbb{R}$ and $\varphi, \Phi \in \mathbb{R}$ with $\Phi > \varphi$. The following statements are equivalent:*

(i) *The function $u - \frac{\varphi+\Phi}{2} \cdot e$, where $e(t) = t$, $t \in [a, b]$, is $\frac{1}{2}(\Phi - \varphi)$ −Lipschitzian;*
(ii) *We have the inequality:*

$$\varphi \le \frac{u(t) - u(s)}{t - s} \le \Phi \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t \ne s; \tag{2.10}$$

(iii) *We have the inequality:*

$$\varphi(t - s) \le u(t) - u(s) \le \Phi(t - s) \quad \text{for each} \quad t, s \in [a, b] \quad \text{with } t > s. \tag{2.11}$$

Following [5], we can introduce the concept:

**Definition 1.** *The function $u : [a, b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be $(\varphi, \Phi)$ −Lipschitzian on $[a, b]$.*

Notice that in [5], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) ⇔ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of $(\varphi, \Phi)$ −Lipschitzian functions.

**Proposition 1.** *Let $u : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. If*

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty \tag{2.12}$$

*then $u$ is $(\gamma, \Gamma)$ −Lipschitzian on $[a, b]$.*

We are able now to provide the following corollary:

**Corollary 1.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a $(\varphi, \Phi)-$Lipschitzian functions on $[m, M]$ with $\Phi > \varphi$, then we have the inequality*

$$\left|\langle f(A)x, y\rangle - \frac{\Phi+\varphi}{2}\langle Ax, y\rangle + \frac{\Phi+\varphi}{2}M\langle x, y\rangle - f(M)\langle x, y\rangle\right| \le \frac{1}{2}(\Phi-\varphi)\int_{m-0}^{M}\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2}dt \le \frac{1}{2}(\Phi-\varphi)\left(M\|x\|^2 - \langle Ax, x\rangle\right)^{1/2}\left(M\|y\|^2 - \langle Ay, y\rangle\right)^{1/2} \tag{2.13}$$

*for any $x, y \in H$.*

Finally, we have the following result for monotonic nondecreasing functions:

**Theorem 3.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then we have the inequality*

$$|f(M)\langle x, y\rangle - \langle f(A)x, y\rangle| \le \int_{m-0}^{M}\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2}df(t) \le \left(f(M)\|x\|^2 - \langle f(A)x, x\rangle\right)^{1/2}\left(f(M)\|y\|^2 - \langle f(A)y, y\rangle\right)^{1/2} \tag{2.14}$$

*for any $x, y \in H$.*

*Proof.* It is well that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral to the identity (2.2) we have

$$|f(M)\langle x, y\rangle - \langle f(A)x, y\rangle| = \left|\int_{m-0}^{M}\langle E_t x, y\rangle\,d(f(t))\right| \le \int_{m-0}^{M}|\langle E_t x, y\rangle|\,d(f(t)) \tag{2.15}$$

for any $x, y \in H$.

By utilizing the generalized Schwarz inequality for nonnegative operators (2.4) and the Cauchy-Buniakovski-Schwarz inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators we have successively

$$\int_{m-0}^{M}|\langle E_t x, y\rangle|\,d(f(t)) \le \int_{m-0}^{M}\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2}d(f(t)) \le \left(\int_{m-0}^{M}\langle E_t x, x\rangle\,d(f(t))\right)^{1/2}\left(\int_{m-0}^{M}\langle E_t y, y\rangle\,d(f(t))\right)^{1/2} \tag{2.16}$$

for any $x, y \in H$.

On the other hand, integrating by parts we also have

$$\int_{m-0}^{M} \langle E_t x, x\rangle \, d(f(t)) = f(t) \langle E_t x, x\rangle|_{m-0}^{M} - \int_{m-0}^{M} f(t) \, d(\langle E_t x, x\rangle) = f(M) \|x\|^2 - \langle f(A) x, x\rangle \tag{2.17}$$

and, similarly,

$$\int_{m-0}^{M} \langle E_t y, y\rangle \, dt = f(M) \|y\|^2 - \langle f(A) y, y\rangle \tag{2.18}$$

for any $x, y \in H$.

On making use of (2.15)-(2.18) we deduce the desired result (2.14). □

**Remark 1.** *Utilising (2.2) we can state the following complementary identity for the continuous function* $f : [m, M] \to \mathbb{C}$

$$\langle f(A) x, y\rangle - f(m) \langle x, y\rangle = \int_{m-0}^{M} \langle (1_H - E_\lambda) x, y\rangle \, d(f(\lambda)) \tag{2.19}$$

*for any* $x, y \in H$.

*On making use of similar arguments provided above we can state the following inequalities as well:*

$$\begin{aligned} &|\langle f(A) x, y\rangle - f(m) \langle x, y\rangle| \\ &\leq \max_{\lambda \in [m,M]} \left[ \langle (1_H - E_\lambda) x, x\rangle^{1/2} \langle (1_H - E_\lambda) y, y\rangle^{1/2} \right] \bigvee_m^M (f) \leq \|x\| \|y\| \bigvee_m^M (f) \end{aligned} \tag{2.20}$$

*if* $f : [m, M] \to \mathbb{C}$ *is of bounded variation on* $[m, M]$;

$$\begin{aligned} &|\langle f(A) x, y\rangle - f(m) \langle x, y\rangle| \\ &\leq L \int_{m-0}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} dt \\ &\leq L \left( \langle Ax, x\rangle - m \|x\|^2 \right)^{1/2} \left( \langle Ay, y\rangle - m \|y\|^2 \right)^{1/2} \end{aligned} \tag{2.21}$$

*if* $f : [m, M] \to \mathbb{C}$ *is Lipschitzian with the constant* $L > 0$ *on* $[m, M]$ *and*

$$\begin{aligned} &|\langle f(A) x, y\rangle - f(m) \langle x, y\rangle| \\ &\leq \int_{m-0}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} df(t) \\ &\leq \left( \langle f(A) x, x\rangle - f(m) \|x\|^2 \right)^{1/2} \left( \langle f(A) y, y\rangle - f(m) \|y\|^2 \right)^{1/2} \end{aligned} \tag{2.22}$$

*if* $f : [m, M] \to \mathbb{R}$ *is monotonic nondecreasing on* $[m, M]$, *where* $x, y \in H$.

## 3. Applications

It is clear that the general inequalities established in the above section can be applied to particular functions of interest. However, we will restrict here only to logarithmic and power functions of operators.

**Proposition 2.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. Then we have the inequalities*

$$\text{(3.1)} \quad |\langle x, y\rangle \ln M - \langle \ln Ax, y\rangle| \leq \max_{\lambda \in [m,M]} \left[ \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \right] \ln \left( \frac{M}{m} \right) \leq \|x\| \|y\| \ln \left( \frac{M}{m} \right),$$

$$\text{(3.2)} \quad |\langle x, y\rangle \ln M - \langle \ln Ax, y\rangle| \leq \frac{1}{m} \int_{m-0}^{M} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} dt \leq \frac{1}{m} \left( M \|x\|^2 - \langle Ax, x\rangle \right)^{1/2} \left( M \|y\|^2 - \langle Ay, y\rangle \right)^{1/2}$$

*and*

$$\text{(3.3)} \quad |\langle x, y\rangle \ln M - \langle \ln Ax, y\rangle| \leq \int_{m-0}^{M} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} t^{-1} dt \leq \left( \|x\|^2 \ln M - \langle \ln Ax, x\rangle \right)^{1/2} \left( \|y\|^2 \ln M - \langle \ln Ay, y\rangle \right)^{1/2}$$

*for any $x, y \in H$.*

The proof follows from Theorems 1, 2 and 3 applied for the function $f : [m, M] \to \mathbb{R}$, $f(t) = \ln t$.

The same theorems applied for the power function will provide:

**Proposition 3.** *Let $A$ be a selfadjoint operator in the Hilbert space $H$ with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $0 < m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. For $p > 0$ we have the inequalities*

$$\text{(3.4)} \quad |M^p \langle x, y\rangle - \langle A^p x, y\rangle| \leq \max_{\lambda \in [m,M]} \left[ \langle E_\lambda x, x\rangle^{1/2} \langle E_\lambda y, y\rangle^{1/2} \right] (M^p - m^p) \leq \|x\| \|y\| (M^p - m^p),$$

$$\text{(3.5)} \quad |M^p \langle x, y\rangle - \langle A^p x, y\rangle| \leq L(p) \int_{m-0}^{M} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} dt \leq L(p) \left( M \|x\|^2 - \langle Ax, x\rangle \right)^{1/2} \left( M \|y\|^2 - \langle Ay, y\rangle \right)^{1/2},$$

*where*

$$L(p) := \begin{cases} pM^{p-1} & \text{if } p \geq 1 \\ pm^{p-1} & \text{if } p \in (0,1) \end{cases}$$

*and*

$$\text{(3.6)} \quad |M^p \langle x, y\rangle - \langle A^p x, y\rangle| \leq p \int_{m-0}^{M} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} t^{p-1} dt \leq \left( \|x\|^2 M^p - \langle A^p x, x\rangle \right)^{1/2} \left( \|y\|^2 M^p - \langle A^p y, y\rangle \right)^{1/2}$$

*for any $x, y \in H$.*

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Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

*E-mail address*: sever.dragomir@vu.edu.au

*URL*: http://www.staff.vu.edu.au/rgmia/dragomir/