

SOME VECTOR INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some inequalities for the composite operator

$$[f(M)1_H - f(A)][f(A) - f(m)1_H],$$

where $Sp(A) \subseteq [m, M]$ and for various classes of continuous functions $f : [m, M] \rightarrow \mathbb{C}$ are given. Applications for the power function and the logarithmic function are also provided.

1. INTRODUCTION

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \rightarrow \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$(1.1) \quad f(U) = \int_{m-0}^M f(\lambda) dE_\lambda,$$

which in terms of vectors can be written as

$$(1.2) \quad \langle f(U)x, y \rangle = \int_{m-0}^M f(\lambda) d\langle E_\lambda x, y \rangle,$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

Utilising the spectral representation from (1.2) we have established the following Ostrowski type vector inequality [5]:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$,*

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then we have the inequality

$$\begin{aligned}
(1.3) \quad & |f(s) \langle x, y \rangle - \langle f(A)x, y \rangle| \\
& \leq \langle E_s x, x \rangle^{1/2} \langle E_s y, y \rangle^{1/2} \bigvee_m^s(f) \\
& + \langle (1_H - E_s)x, x \rangle^{1/2} \langle (1_H - E_s)y, y \rangle^{1/2} \bigvee_s^M(f) \\
& \leq \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M(f) + \frac{1}{2} \left| \bigvee_m^s(f) - \bigvee_s^M(f) \right| \right) \leq \|x\| \|y\| \bigvee_m^M(f)
\end{aligned}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

Another result that compares the function of a selfadjoint operator with the integral mean is embodied in the following theorem [6]:

Theorem 2. *With the assumptions in Theorem 1 we have the inequalities*

$$\begin{aligned}
(1.4) \quad & \left| \langle x, y \rangle \cdot \frac{1}{M-m} \int_m^M f(s) ds - \langle f(A)x, y \rangle \right| \\
& \leq \bigvee_m^M(f) \max_{t \in [m, M]} \left[\frac{M-t}{M-m} \langle E_t x, x \rangle^{1/2} \langle E_t y, y \rangle^{1/2} \right. \\
& \quad \left. + \frac{t-m}{M-m} \langle (1_H - E_t)x, x \rangle^{1/2} \langle (1_H - E_t)y, y \rangle^{1/2} \right] \\
& \leq \|x\| \|y\| \bigvee_m^M(f)
\end{aligned}$$

for any $x, y \in H$.

The trapezoid version of the above result has been obtained in [7] and is as follows:

Theorem 3. *With the assumptions in Theorem 1 we have the inequalities*

$$\begin{aligned}
(1.5) \quad & \left| \frac{f(M) + f(m)}{2} \cdot \langle x, y \rangle - \langle f(A)x, y \rangle \right| \\
& \leq \frac{1}{2} \max_{\lambda \in [m, M]} \left[\langle E_\lambda x, x \rangle^{1/2} \langle E_\lambda y, y \rangle^{1/2} \right. \\
& \quad \left. + \langle (1_H - E_\lambda)x, x \rangle^{1/2} \langle (1_H - E_\lambda)y, y \rangle^{1/2} \right] \bigvee_m^M(f) \\
& \leq \frac{1}{2} \|x\| \|y\| \bigvee_m^M(f)
\end{aligned}$$

for any $x, y \in H$.

For various inequalities for functions of selfadjoint operators, see [8]-[12]. For recent results see [1]-[5].

In this paper we investigate the quantity

$$|\langle [f(M)1_H - f(A)][f(A) - f(m)1_H]x, y \rangle|$$

where x, y are vectors in the Hilbert space H and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$, and provide different bounds for some classes of continuous functions $f : [m, M] \rightarrow \mathbb{C}$. Applications for some particular cases including the power and logarithmic functions are provided as well.

2. SOME VECTOR INEQUALITIES

The following representation in terms of the spectral family is of interest in itself:

Lemma 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{C}$ is a continuous function on $[m, M]$ with $f(M) \neq f(m)$, then we have the representation*

$$(2.1) \quad \begin{aligned} & \frac{1}{[f(M) - f(m)]^2} [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] \\ &= \frac{1}{f(M) - f(m)} \\ & \quad \times \int_{m-0}^M \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) \left(E_t - \frac{1}{2} 1_H \right) df(t). \end{aligned}$$

Proof. We observe that,

$$(2.2) \quad \begin{aligned} & \frac{1}{f(M) - f(m)} \int_{m-0}^M \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) \\ & \quad \times \left(E_t - \frac{1}{2} 1_H \right) df(t) \\ &= \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t^2 df(t) \\ & \quad - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \\ & \quad - \frac{1}{2} \int_{m-0}^M E_t df(t) + \frac{1}{2} \int_{m-0}^M E_s df(s) \\ &= \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t^2 df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \right]^2 \end{aligned}$$

which is an equality of interest in itself.

Since E_t are projections, we have $E_t^2 = E_t$ for any $t \in [m, M]$ and then we can write that

$$(2.3) \quad \begin{aligned} & \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t^2 df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \right]^2 \\ &= \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) - \left[\frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \right]^2 \\ &= \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \left[1_H - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) \right]. \end{aligned}$$

Integrating by parts in the Riemann-Stieltjes integral and utilizing the spectral representation (1.1) we have

$$\int_{m-0}^M E_t df(t) = f(M) 1_H - f(A)$$

and

$$1_H - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_t df(t) = \frac{f(A) - f(m) 1_H}{f(M) - f(m)},$$

which together with (2.3) and (2.2) produce the desired result (2.1). \square

The following vector version may be stated as well:

Corollary 1. *With the assumptions of Lemma 1 we have the equality*

$$(2.4) \quad \begin{aligned} & \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle \\ &= [f(M) - f(m)] \\ & \times \int_{m-0}^M \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle df(t), \end{aligned}$$

for any $x, y \in [m, M]$.

The following result that provides some bounds for continuous functions of bounded variation may be stated as well:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ with $f(M) \neq f(m)$, then we have the inequality*

$$(2.5) \quad \begin{aligned} & | \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle | \\ & \leq \frac{1}{2} \|y\| |f(M) - f(m)| \bigvee_m^M(f) \\ & \times \sup_{t \in [m, M]} \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right\| \\ & \leq \frac{1}{2} \|x\| \|y\| \left[\bigvee_m^M(f) \right]^2, \end{aligned}$$

for any $x, y \in H$.

Proof. It is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is a bounded function, $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then the following inequality holds

$$(2.6) \quad \left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Utilising this property and the representation (2.4) we have by the Schwarz inequality in Hilbert space H that

$$\begin{aligned}
(2.7) \quad & | \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle | \\
& \leq |f(M) - f(m)| \bigvee_m^M (f) \\
& \times \sup_{t \in [m, M]} \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle \right| \\
& \leq |f(M) - f(m)| \bigvee_m^M (f) \\
& \times \sup_{t \in [m, M]} \left[\left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| \right]
\end{aligned}$$

for any $x, y \in [m, M]$.

Since E_t are projections, and in this case we have

$$\begin{aligned}
\left\| E_t y - \frac{1}{2} y \right\|^2 &= \|E_t y\|^2 - \langle E_t y, y \rangle + \frac{1}{4} \|y\|^2 \\
&= \langle E_t^2 y, y \rangle - \langle E_t y, y \rangle + \frac{1}{4} \|y\|^2 = \frac{1}{4} \|y\|^2,
\end{aligned}$$

then from (2.7) we deduce the first part of (2.5).

Now, by the same property (2.6) for vector valued functions p with values in Hilbert spaces, we also have that

$$\begin{aligned}
(2.8) \quad & \left\| [f(M) - f(m)] E_t x - \int_{m-0}^M E_s x df(s) \right\| \\
&= \left\| \int_{m-0}^M (E_t x - E_s x) df(s) \right\| \leq \bigvee_m^M (f) \sup_{s \in [m, M]} \|E_t x - E_s x\|
\end{aligned}$$

for any $t \in [m, M]$ and $x \in H$.

Since $0 \leq E_t \leq 1_H$ in the operator order, then $-1_H \leq E_t - E_s \leq 1$ which gives that $-\|x\|^2 \leq \langle (E_t - E_s)x, x \rangle \leq \|x\|^2$, i.e., $|\langle (E_t - E_s)x, x \rangle| \leq \|x\|^2$ for any $x \in H$, which implies that $\|E_t - E_s\| \leq 1$ for any $t, s \in [m, M]$. Therefore $\sup_{s \in [m, M]} \|E_t x - E_s x\| \leq \|x\|$ which together with (2.8) prove the last part of (2.5). \square

The case of Lipschitzian functions is as follows:

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on*

$[m, M]$ and with $f(M) \neq f(m)$, then we have the inequality

$$\begin{aligned}
(2.9) \quad & |\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle| \\
& \leq \frac{1}{2} L \|y\| |f(M) - f(m)| \\
& \quad \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| dt \\
& \leq \frac{1}{2} L^2 \|y\| \int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\| ds dt \\
& \leq \frac{\sqrt{2}}{2} L^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2} \\
& \leq \frac{\sqrt{2}}{4} L^2 \|y\| \|x\| (M - m)^2
\end{aligned}$$

for any $x, y \in H$.

Proof. Recall that if $p : [a, b] \rightarrow \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists and the following inequality holds

$$(2.10) \quad \left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| dt.$$

Now, on applying this property of the Riemann-Stieltjes integral, then we have from the representation (2.4) that

$$\begin{aligned}
(2.11) \quad & |\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle| \\
& \leq |f(M) - f(m)| \\
& \quad \times \int_{m-0}^M \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) x, \left(E_t - \frac{1}{2} 1_H \right) y \right\rangle \right| df(t), \\
& \leq L |f(M) - f(m)| \\
& \quad \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| dt \\
& = \frac{1}{2} L \|y\| |f(M) - f(m)| \\
& \quad \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| dt
\end{aligned}$$

for any $x, y \in H$ and the first inequality in (2.9) is proved.

Further, observe that

$$\begin{aligned}
(2.12) \quad & |f(M) - f(m)| \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| dt \\
&= \int_{m-0}^M \left\| [f(M) - f(m)] E_t x - \int_{m-0}^M E_s x df(s) \right\| dt \\
&= \int_{m-0}^M \left\| \int_{m-0}^M (E_t x - E_s x) df(s) \right\| dt
\end{aligned}$$

for any $x \in H$.

If we use the vector valued version of the property (2.10), then we have

$$(2.13) \quad \int_{m-0}^M \left\| \int_{m-0}^M (E_t x - E_s x) df(s) \right\| dt \leq L \int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\| ds dt$$

for any $x \in H$ and the second part of (2.9) is proved.

Further on, by applying the double integral version of the Cauchy-Buniakowski-Schwarz inequality we have

$$\begin{aligned}
(2.14) \quad & \int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\| ds dt \\
& \leq (M - m) \left(\int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\|^2 ds dt \right)^{1/2}
\end{aligned}$$

for any $x \in H$.

Now, by utilizing the fact that E_s are projections for each $s \in [m, M]$, then we have

$$\begin{aligned}
(2.15) \quad & \int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\|^2 ds dt \\
&= 2 \left[(M - m) \int_{m-0}^M \|E_t x\|^2 dt - \left\| \int_{m-0}^M E_t x dt \right\|^2 \right] \\
&= 2 \left[(M - m) \int_{m-0}^M \langle E_t x, x \rangle dt - \left\| \int_{m-0}^M E_t x dt \right\|^2 \right]
\end{aligned}$$

for any $x \in H$.

If we integrate by parts and use the spectral representation (1.2), then we get

$$\int_{m-0}^M \langle E_t x, x \rangle dt = \langle Mx - Ax, x \rangle \quad \text{and} \quad \int_{m-0}^M E_t x dt = Mx - Ax$$

and by (2.15) we then obtain the following equality of interest

$$(2.16) \quad \int_{m-0}^M \int_{m-0}^M \|E_t x - E_s x\|^2 ds dt = 2 \langle Ax - mx, Mx - Ax \rangle$$

for any $x \in H$.

On making use of (2.16) and (2.14) we then deduce the third part of (2.9).

Finally, by utilizing the elementary inequality in inner product spaces

$$(2.17) \quad \operatorname{Re} \langle a, b \rangle \leq \frac{1}{4} \|a + b\|^2, \quad a, b \in H,$$

we also have that

$$\langle Ax - mx, Mx - Ax \rangle \leq \frac{1}{4} (M - m)^2 \|x\|^2$$

for any $x \in H$, which proves the last inequality in (2.9). \square

The case of nondecreasing monotonic functions is as follows:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \rightarrow \mathbb{R}$ is a monotonic nondecreasing function on $[m, M]$, then we have the inequality*

$$(2.18) \quad \begin{aligned} & |\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, y \rangle| \\ & \leq \frac{1}{2} \|y\| [f(M) - f(m)] \\ & \quad \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| df(t) \\ & \leq \frac{1}{2} \|y\| [f(M) - f(m)] \\ & \quad \times \langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, x \rangle^{1/2} \\ & \leq \frac{1}{4} \|y\| \|x\| [f(M) - f(m)]^2 \end{aligned}$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral it is also well known that if $p : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and $v : [a, b] \rightarrow \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_a^b p(t) dv(t)$ and $\int_a^b |p(t)| dv(t)$ exist and

$$\left| \int_a^b p(t) dv(t) \right| \leq \int_a^b |p(t)| dv(t).$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.4) that

(2.19)

$$\begin{aligned}
& | \langle [f(M) \mathbf{1}_H - f(A)] [f(A) - f(m) \mathbf{1}_H] x, y \rangle | \\
& \leq [f(M) - f(m)] \\
& \times \int_{m-0}^M \left| \left\langle \left(E_t - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s df(s) \right) x, \left(E_t - \frac{1}{2} \mathbf{1}_H \right) y \right\rangle \right| df(t), \\
& \leq [f(M) - f(m)] \\
& \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| \left\| E_t y - \frac{1}{2} y \right\| df(t) \\
& = \frac{1}{2} \|y\| [f(M) - f(m)] \\
& \times \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| df(t)
\end{aligned}$$

for any $x, y \in H$, which proves the first inequality in (2.18).

On utilizing the Cauchy-Buniakowski-Schwarz type inequality for the Riemann-Stieltjes integral of monotonic nondecreasing integrators, we have

$$\begin{aligned}
(2.20) \quad & \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\| df(t) \\
& \leq \left[\int_{m-0}^M df(t) \right]^{1/2} \\
& \times \left[\int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 df(t) \right]^{1/2}
\end{aligned}$$

for any $x, y \in H$.

Observe that

$$\begin{aligned}
(2.21) \quad & \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 df(t) \\
& = \int_{m-0}^M \left[\|E_t x\|^2 - 2 \operatorname{Re} \left\langle E_t x, \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\rangle \right. \\
& \quad \left. + \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 \right] df(t) \\
& = [f(M) - f(m)] \left[\frac{1}{f(M) - f(m)} \int_{m-0}^M \|E_t x\|^2 df(t) \right. \\
& \quad \left. - \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 \right]
\end{aligned}$$

and, integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
(2.22) \quad \int_{m-0}^M \|E_t x\|^2 df(t) &= \int_{m-0}^M \langle E_t x, E_t x \rangle df(t) = \int_{m-0}^M \langle E_t x, x \rangle df(t) \\
&= f(M) \|x\|^2 - \int_{m-0}^M f(t) d \langle E_t x, x \rangle \\
&= f(M) \|x\|^2 - \langle f(A) x, x \rangle = \langle [f(M) 1_H - f(A)] x, x \rangle
\end{aligned}$$

and

$$(2.23) \quad \int_{m-0}^M E_s x df(s) = f(M) x - f(A) x$$

for any $x \in H$.

On making use of the equalities (2.22) and (2.23) we have

$$\begin{aligned}
(2.24) \quad & \frac{1}{f(M) - f(m)} \int_{m-0}^M \|E_t x\|^2 df(t) - \left\| \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 \\
&= \frac{1}{[f(M) - f(m)]^2} \\
&\times \left[[f(M) - f(m)] \langle [f(M) 1_H - f(A)] x, x \rangle - \|f(M) x - f(A) x\|^2 \right] \\
&= \frac{[f(M) - f(m)] \langle [f(M) 1_H - f(A)] x, x \rangle - \langle f(M) x - f(A) x, f(M) x - f(A) x \rangle}{[f(M) - f(m)]^2} \\
&= \frac{[f(M) - f(m)] \langle [f(M) 1_H - f(A)] x, x \rangle - \langle f(M) x - f(A) x, f(M) x - f(A) x \rangle}{[f(M) - f(m)]^2} \\
&= \frac{\langle f(M) x - f(A) x, f(A) x - f(m) x \rangle}{[f(M) - f(m)]^2}
\end{aligned}$$

for any $x \in H$.

Therefore, we obtain the following equality of interest in itself as well

$$\begin{aligned}
(2.25) \quad & \frac{1}{f(M) - f(m)} \int_{m-0}^M \left\| E_t x - \frac{1}{f(M) - f(m)} \int_{m-0}^M E_s x df(s) \right\|^2 df(t) \\
&= \frac{\langle f(M) x - f(A) x, f(A) x - f(m) x \rangle}{[f(M) - f(m)]^2} \\
&= \frac{\langle [f(M) 1_H - f(A)] [f(A) - f(m) 1_H] x, x \rangle}{[f(M) - f(m)]^2}
\end{aligned}$$

for any $x \in H$

On making use of the inequality (2.20) we deduce the second inequality in (2.18).

The last part follows by (2.17) and the details are omitted. \square

3. APPLICATIONS

We consider the power function $f(t) := t^p$ where $p \in \mathbb{R} \setminus \{0\}$ and $t > 0$. The following power inequalities hold:

Proposition 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$.*

If $p > 0$, then for any $x, y \in H$

$$(3.1) \quad \begin{aligned} & |\langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, y \rangle| \\ & \leq \frac{\sqrt{2}}{2} B_p^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2} \\ & \leq \frac{\sqrt{2}}{4} B_p^2 \|y\| \|x\| (M - m)^2 \end{aligned}$$

where

$$B_p = p \times \begin{cases} M^{p-1} & \text{if } p \geq 1 \\ m^{p-1} & \text{if } 0 < p < 1, m > 0 \end{cases}$$

and

$$(3.2) \quad \begin{aligned} & |\langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p})x, y \rangle| \\ & \leq \frac{\sqrt{2}}{2} C_p^2 \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2} \\ & \leq \frac{\sqrt{2}}{4} C_p^2 \|y\| \|x\| (M - m)^2, \end{aligned}$$

where

$$C_p = pm^{-p-1} \text{ and } m > 0.$$

The proof follows from Theorem 5 applied for the power function.

Proposition 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 \leq m < M$.*

If $p > 0$, then for any $x, y \in H$

$$(3.3) \quad \begin{aligned} & |\langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, y \rangle| \\ & \leq \frac{1}{2} \|y\| (M^p - m^p) \langle (M^p 1_H - A^p)(A^p - m^p 1_H)x, x \rangle^{1/2} \\ & \leq \frac{1}{4} \|y\| \|x\| (M^p - m^p)^2 \end{aligned}$$

and

$$(3.4) \quad \begin{aligned} & |\langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p})x, y \rangle| \\ & \leq \frac{1}{2} \|y\| (m^{-p} - M^{-p}) \langle (A^{-p} - M^{-p} 1_H)(m^{-p} 1_H - A^{-p})x, x \rangle^{1/2} \\ & \leq \frac{1}{4} \|y\| \|x\| (m^{-p} - M^{-p})^2. \end{aligned}$$

The proof follows from Theorem 6.

Now, consider the logarithmic function $f(t) = \ln t, t > 0$. We have

Proposition 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers with $0 < m < M$. Then we have*

the inequalities

$$\begin{aligned}
 (3.5) \quad & | \langle [(\ln M) 1_H - \ln A] [\ln A - (\ln m) 1_H] x, y \rangle | \\
 & \leq \frac{\sqrt{2}}{2m^2} \|y\| (M - m) \langle Ax - mx, Mx - Ax \rangle^{1/2} \\
 & \leq \frac{\sqrt{2}}{4} \|y\| \|x\| \left(\frac{M}{m} - 1 \right)^2
 \end{aligned}$$

and

$$\begin{aligned}
 (3.6) \quad & | \langle [(\ln M) 1_H - \ln A] [\ln A - (\ln m) 1_H] x, y \rangle | \\
 & \leq \frac{1}{2} \|y\| \langle [(\ln M) 1_H - \ln A] [\ln A - (\ln m) 1_H] x, x \rangle^{1/2} \ln \left(\frac{M}{m} \right) \\
 & \leq \frac{1}{4} \|y\| \|x\| \left[\ln \left(\frac{M}{m} \right) \right]^2.
 \end{aligned}$$

The proof follows from Theorem 5 and 6 applied for the logarithmic function.

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