ERROR BOUNDS IN APPROXIMATING n-TIME DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES VIA A TAYLOR'S TYPE EXPANSION

S.S. DRAGOMIR

ABSTRACT. On utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some error bounds in approximating n-time differentiable functions of selfadjoint operators in Hilbert Spaces via a Taylor's type expansion are given.

1. INTRODUCTION

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda, \tag{1.1}$$

which in terms of vectors can be written as

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle, \tag{1.2}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [9] and the references therein.

For other recent results see [1], [2], [3], [8], [11], [12], [13] and [14].

Utilising the spectral representation from (1.2) we have established the following Ostrowski type vector inequality [5]:

Theorem 1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral*

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Functions of Selfadjoint operators, Spectral representation, Inequalities for selfadjoint operators.

family. If $f:[m,M]\to\mathbb{C}$ *is a continuous function of bounded variation on* $[m,M]$*, then we have the inequality*

$$|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \tag{1.3}$$
$$\leq\langle E_s x,x\rangle^{1/2}\langle E_s y,y\rangle^{1/2}\bigvee_m^s(f)$$
$$+\langle(1_H-E_s)x,x\rangle^{1/2}\langle(1_H-E_s)y,y\rangle^{1/2}\bigvee_s^M(f)$$
$$\leq\|x\|\|y\|\left(\frac{1}{2}\bigvee_m^M(f)+\frac{1}{2}\left|\bigvee_m^s(f)-\bigvee_s^M(f)\right|\right)$$
$$\leq\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$ *and for any* $s\in[m,M]$.

Another result that compares the function of a selfadjoint operator with the integral mean is embodied in the following theorem [6]:

Theorem 2. *With the assumptions in Theorem 1 we have the inequalities*

$$\left|\langle x,y\rangle\cdot\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{1.4}$$
$$\leq\frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big]$$
$$\leq\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

The trapezoid version of the above result has been obtained in [4] and is as follows:

Theorem 3. *With the assumptions in Theorem 1 we have the inequalities*

$$\left|\frac{f(M)+f(m)}{2}\cdot\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.5}$$
$$\leq\frac{1}{2}\max_{\lambda\in[m,M]}\Big[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$
$$+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\Big]\bigvee_m^M(f)$$
$$\leq\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

A generalized trapezoid type inequality was obtained in [7] and is as follows:

Theorem 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family.*

1. If $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation on $[m, M]$, then

$$\left|\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right] x, y \right\rangle - \langle f(A)x, y\rangle\right| \tag{1.6}$$
$$\leq \sup_{t\in[m,M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right] \bigvee_{m}^{M}(f)$$
$$\leq \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \bigvee_{m}^{M}(f) \leq \|x\| \|y\| \bigvee_{m}^{M}(f)$$

for any $x, y \in H$.

2. If $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\left|\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right] x, y \right\rangle - \langle f(A)x, y\rangle\right| \tag{1.7}$$
$$\leq L \int_{m-0}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right] dt$$
$$\leq L(M-m) \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq L(M-m)\|x\| \|y\|$$

for any $x, y \in H$.

3. If $f : [m, M] \to \mathbb{R}$ is continuous and monotonic nondecreasing on $[m, M]$, then

$$\left|\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m}\right] x, y \right\rangle - \langle f(A)x, y\rangle\right| \tag{1.8}$$
$$\leq \int_{m-0}^{M} \left[\frac{t-m}{M-m} \bigvee_{m-0}^{t} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + \frac{M-t}{M-m} \bigvee_{t}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right] df(t)$$
$$\leq \bigvee_{m-0}^{M} \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) [f(M) - f(m)] \leq \|x\| \|y\| [f(M) - f(m)]$$

for any $x, y \in H$.

In this paper, by utilizing the spectral representation of selfadjoint operators in Hilbert spaces, some error bounds in approximating n-time differentiable functions of selfadjoint operators in Hilbert Spaces via a Taylor's type expansion are given. Applications for some elementary functions of interest including the exponential and logarithmic functions are also provided.

2. Some Identities

The following result provides a Taylor's type representation for a function of selfadjoint operators in Hilbert spaces with integral remainder.

Theorem 5. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family,* *I be a closed subinterval on* $\mathbb{R}$ *with* $[m,M]\subset \mathring{I}$ *(the interior of* I*) and let* n *be an integer with* $n\geq 1$*. If* $f:I\to\mathbb{C}$ *is such that the* n*-th derivative* $f^{(n)}$ *is of bounded variation on the interval* $[m,M]$*, then for any* $c\in[m,M]$ *we have the equalities*

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(A-c1_H)^k+R_n(f,c,m,M) \tag{2.1}$$

where

$$R_n(f,c,m,M)=\frac{1}{n!}\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda. \tag{2.2}$$

Proof. We utilize the Taylor formula for a function $f:I\to\mathbb{C}$ whose n-th derivative $f^{(n)}$ is locally of bounded variation on the interval I to write the equality

$$f(\lambda)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)(\lambda-c)^k+\frac{1}{n!}\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right) \tag{2.3}$$

for any $\lambda,c\in[m,M]$, where the integral is taken in the Riemann-Stieltjes sense.

If we integrate the equality on $[m,M]$ in the Riemann-Stieltjes sense with the integrator E_λ we get

$$\begin{aligned}\int_{m-0}^{M}f(\lambda)\,dE_\lambda&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\int_{m-0}^{M}(\lambda-c)^k dE_\lambda\\&+\frac{1}{n!}\int_{m-0}^{M}\left(\int_c^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda\end{aligned}$$

which, by the spectral representation (1.1), produces the equality (2.1) with the representation of the remainder from (2.2). □

The following particular instances are of interest for applications:

Corollary 1. *With the assumptions of the above Theorem 5, we have the equalities*

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)(A-m1_H)^k+L_n(f,c,m,M) \tag{2.4}$$

where

$$L_n(f,c,m,M)=\frac{1}{n!}\int_{m-0}^{M}\left(\int_m^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda$$

and

$$f(A)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left(A-\frac{m+M}{2}1_H\right)^k+M_n(f,c,m,M) \tag{2.5}$$

where

$$M_n(f,c,m,M)=\frac{1}{n!}\int_{m-0}^{M}\left(\int_{\frac{m+M}{2}}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)dE_\lambda$$

and

$$f(A)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(M)(M1_H-A)^k+U_n(f,c,m,M) \tag{2.6}$$

where

$$U_n(f,c,m,M)=\frac{(-1)^{n+1}}{n!}\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-\lambda)^n\,d\left(f^{(n)}(t)\right)\right)dE_\lambda, \tag{2.7}$$

respectively.

Remark 1. *We remark that, if the n-th derivative of the function f considered above is absolutely continuous on the interval $[m,M]$, then we have the representation (2.1) with the remainder*

$$R_n(f,c,m,M)=\frac{1}{n!}\int_{m-0}^{M}\left(\int_{c}^{\lambda}(\lambda-t)^n f^{(n+1)}(t)\,dt\right)dE_\lambda. \tag{2.8}$$

Here the integral $\int_c^\lambda (\lambda-t)^n f^{(n+1)}(t)\,dt$ is considered in the Lebesgue sense. Similar representations hold true when c is taken the particular values m, M or $\frac{m+M}{2}$.

Now, if we consider the exponential function, then for any selfadjoint operator A in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ and with the spectral family $\{E_\lambda\}_\lambda$ we have the representation

$$e^{A-c1_H}=\sum_{k=0}^{n}\frac{1}{k!}(A-c1_H)^k+\frac{1}{n!}\int_{m-0}^{M}\left(\int_{c}^{\lambda}(\lambda-t)^n e^{t-c}dt\right)dE_\lambda, \tag{2.9}$$

where c is any real number.

Further, if we consider the logarithmic function, then for any positive definite operator A with $Sp(A)\subseteq[m,M]\subset(0,\infty)$ and with the spectral family $\{E_\lambda\}_\lambda$ we have

$$\ln A=(\ln c)\,1_H+\sum_{k=1}^{n}\frac{(-1)^{k-1}(A-c1_H)^k}{kc^k} \tag{2.10}$$
$$+(-1)^n\int_{m-0}^{M}\left(\int_{c}^{\lambda}\frac{(\lambda-t)^n}{t^{n+1}}dt\right)dE_\lambda$$

for any $c>0$.

3. Some Error Bounds

We start with the following result that provides an approximation for an n-time differentiable function of selfadjoint operators in Hilbert spaces:

Theorem 6. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M]\subset\mathring{I}$ (the interior of I) and let n be an integer with $n\geq1$. If $f:I\to\mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded*

variation on the interval $[m,M]$*, then for any* $c\in[m,M]$ *we have the inequality*

$$
\begin{aligned}
&|\langle R_n(f,c,m,M)x,y\rangle| \\
&=\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle\right| \\
&\leq\frac{1}{n!}\left[(c-m)^n\bigvee_m^c\left(f^{(n)}\right)\bigvee_m^c\left(\langle E_{(\cdot)}x,y\rangle\right)\right. \\
&\left.+(M-c)^n\bigvee_c^M\left(f^{(n)}\right)\bigvee_c^M\left(\langle E_{(\cdot)}x,y\rangle\right)\right] \\
&\leq\frac{1}{n!}\max\left\{(M-c)^n\bigvee_c^M\left(f^{(n)}\right),(c-m)^n\bigvee_c^M\left(f^{(n)}\right)\right\}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&\leq\frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n\bigvee_m^M\left(f^{(n)}\right)\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right),
\end{aligned}
\tag{3.1}
$$

for any $x,y\in H$.

Proof. From the identities (2.1) and (2.2) we have

$$
\begin{aligned}
&\langle R_n(f,c,m,M)x,y\rangle \\
&=\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(c)\left\langle (A-c1_H)^k x,y\right\rangle \\
&=\frac{1}{n!}\int_{m-0}^{M}\left(\int_c^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle \\
&=\frac{1}{n!}\int_{m-0}^{c}\left(\int_c^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle \\
&+\frac{1}{n!}\int_{c}^{M}\left(\int_c^\lambda(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle
\end{aligned}
\tag{3.2}
$$

for any $x,y\in H$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function, $v:[a,b]\to\mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$
\left|\int_a^b p(t)\,dv(t)\right|\leq\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),
\tag{3.3}
$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Taking the modulus in (3.2) and utilizing the inequality (3.3) we have

$$\begin{aligned}
&|\langle R_n(f,c,m,M)x,y\rangle| \\
&\leq \frac{1}{n!}\left|\int_{m-0}^{c}\left(\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\right| \\
&+\frac{1}{n!}\left|\int_{c}^{M}\left(\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right)d\langle E_\lambda x,y\rangle\right| \\
&\leq \frac{1}{n!}\max_{\lambda\in[m,c]}\left|\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\bigvee_{m}^{c}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&+\frac{1}{n!}\max_{\lambda\in[c,M]}\left|\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\bigvee_{c}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)
\end{aligned} \tag{3.4}$$

for any $x,y\in H$.

By the same property (3.3) we have

$$\max_{\lambda\in[m,c]}\left|\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\leq (c-m)^n\bigvee_{m}^{c}\left(f^{(n)}\right) \tag{3.5}$$

and

$$\max_{\lambda\in[c,M]}\left|\int_{c}^{\lambda}(\lambda-t)^n d\left(f^{(n)}(t)\right)\right|\leq (M-c)^n\bigvee_{c}^{M}\left(f^{(n)}\right). \tag{3.6}$$

Now, on making use of (3.4)-(3.6) we deduce

$$\begin{aligned}
&|\langle R_n(f,c,m,M)x,y\rangle| \\
&\leq \frac{1}{n!}\left[(c-m)^n\bigvee_{m}^{c}\left(f^{(n)}\right)\bigvee_{m}^{c}\left(\langle E_{(\cdot)}x,y\rangle\right)\right. \\
&\left.+(M-c)^n\bigvee_{c}^{M}\left(f^{(n)}\right)\bigvee_{c}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n\bigvee_{m}^{c}\left(f^{(n)}\right),(M-c)^n\bigvee_{c}^{M}\left(f^{(n)}\right)\right\} \\
&\times\left[\bigvee_{m}^{c}\left(\langle E_{(\cdot)}x,y\rangle\right)+\bigvee_{c}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)\right] \\
&\leq \frac{1}{n!}\max\left\{(c-m)^n,(M-c)^n\right\}\bigvee_{m}^{M}\left(f^{(n)}\right)\bigvee_{m}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&=\frac{1}{n!}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^n\bigvee_{m}^{M}\left(f^{(n)}\right)\bigvee_{m}^{M}\left(\langle E_{(\cdot)}x,y\rangle\right)
\end{aligned}$$

for any $x,y\in H$ and the proof is complete. □

The following particular cases are of interest for applications

Corollary 2. *With the assumption of Theorem 6 we have the inequalities*

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(m)\left\langle (A-m1_H)^k x,y\right\rangle\right| \leq \frac{1}{n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \leq \frac{1}{n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\|x\|\,\|y\|, \tag{3.7}$$

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(M)\left\langle (M1_H-A)^k x,y\right\rangle\right| \leq \frac{1}{n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \leq \frac{1}{n!}(M-m)^n\bigvee_m^M\left(f^{(n)}\right)\|x\|\,\|y\| \tag{3.8}$$

and

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left\langle \left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\right| \leq \frac{1}{2^n n!}(M-m)^n\max\left\{\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right),\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\right\}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \leq \frac{1}{2^n n!}(M-m)^n\max\left\{\bigvee_{\frac{m+M}{2}}^{M}\left(f^{(n)}\right),\bigvee_{m}^{\frac{m+M}{2}}\left(f^{(n)}\right)\right\}\|x\|\,\|y\| \tag{3.9}$$

respectively, for any $x,y\in H$.

Proof. The first part in the inequalities follow from (3.1) by choosing $c=m, c=M$ and $c=\frac{m+M}{2}$ respectively.

If P is a nonnegative operator on H, i.e., $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle \tag{3.10}$$

for any $x,y\in H$.

Now, if $d: m=t_0<t_1<...<t_{n-1}<t_n=M$ is an arbitrary partition of the interval $[m,M]$, then we have by Schwarz's inequality for nonnegative operators

(3.10) that

$$\begin{aligned}
&\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&= \sup_d \left\{ \sum_{i=0}^{n-1} \left| \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y \right\rangle \right| \right\} \\
&\leq \sup_d \left\{ \sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle^{1/2} \right] \right\} := B.
\end{aligned}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}
B &\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&\leq \sup_d \left\{ \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x \right\rangle \right]^{1/2} \sup_d \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y \right\rangle \right]^{1/2} \right\} \\
&= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}x, x\right\rangle\right) \right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}y, y\right\rangle\right) \right]^{1/2} = \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$. These prove the last part of the above inequalities (3.7)-(3.9). □

The following result also holds:

Theorem 7. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$\begin{aligned}
&|\langle R_n(f, c, m, M)x, y\rangle| \\
&\leq \frac{1}{(n+1)!} L_n \left[(c-m)^{n+1} \bigvee_m^c \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + (M-c)^{n+1} \bigvee_c^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \right] \\
&\leq \frac{1}{(n+1)!} L_n \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\
&\leq \frac{1}{(n+1)!} L_n \left(\frac{1}{2}(M-m) + \left| c - \frac{m+M}{2} \right| \right)^{n+1} \|x\| \|y\|
\end{aligned} \tag{3.11}$$

for any $x, y \in H$.

Proof. First of all, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann-Stieltjes integral we have

$$\max_{\lambda\in[m,c]}\left|\int_\lambda^c (t-\lambda)^n\, d\left(f^{(n)}(t)\right)\right| \le \max_{\lambda\in[m,c]}\left[L_n\int_\lambda^c (t-\lambda)^n\,dt\right] = \frac{L_n}{n+1}(c-m)^{n+1} \tag{3.12}$$

and

$$\max_{\lambda\in[c,M]}\left|\int_c^\lambda (\lambda-t)^n\, d\left(f^{(n)}(t)\right)\right| \le \max_{\lambda\in[c,M]}\left[L_n\int_c^\lambda (\lambda-t)^n\,dt\right] = \frac{L_n}{n+1}(M-c)^{n+1}. \tag{3.13}$$

Now, on utilizing the inequality (3.4), then we have from (3.12) and (3.13) that

$$\begin{aligned}
&|\langle R_n(f,c,m,M)x,y\rangle| \\
&\le \frac{1}{(n+1)!}L_n(c-m)^{n+1}\bigvee_m^c\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&+\frac{1}{(n+1)!}L_n(M-c)^{n+1}\bigvee_c^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&\le \frac{1}{(n+1)!}L_n\max\left\{(c-m)^{n+1},(M-c)^{n+1}\right\}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&= \frac{1}{(n+1)!}L_n\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right),
\end{aligned} \tag{3.14}$$

and the proof is complete. □

The following particular cases are of interest for applications:

Corollary 3. *With the assumption of Theorem 7 we have the inequalities*

$$\begin{aligned}
&\left|\langle f(A)x,y\rangle-\sum_{k=0}^n\frac{1}{k!}f^{(k)}(m)\left\langle (A-m1_H)^k x,y\right\rangle\right| \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \\
&\le \frac{1}{(n+1)!}(M-m)^{n+1}L_n\|x\|\,\|y\|
\end{aligned} \tag{3.15}$$

and

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}f^{(k)}(M)\left\langle(M1_{H}-A)^{k}x,y\right\rangle\right| \tag{3.16}$$
$$\leq\frac{1}{(n+1)!}(M-m)^{n+1}L_{n}\bigvee_{m}^{M}(\langle E_{(\cdot)}x,y\rangle)$$
$$\leq\frac{1}{(n+1)!}(M-m)^{n+1}L_{n}\|x\|\|y\|$$

and

$$\left|\langle f(A)x,y\rangle-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_{H}\right)^{k}x,y\right\rangle\right| \tag{3.17}$$
$$\leq\frac{1}{2^{n+1}(n+1)!}(M-m)^{n+1}L_{n}\bigvee_{m}^{M}(\langle E_{(\cdot)}x,y\rangle)$$
$$\leq\frac{1}{2^{n+1}(n+1)!}(M-m)^{n+1}L_{n}\|x\|\|y\|$$

respectively, for any $x,y\in H$.

The following lemma may be stated.

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ and $\varphi,\Phi\in\mathbb{R}$ be such that $\Phi>\varphi$. The following statements are equivalent:*

(i) *The function $u-\frac{\varphi+\Phi}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Phi-\varphi)-$Lipschitzian;*

(ii) *We have the inequality:*

$$\varphi\leq\frac{u(t)-u(s)}{t-s}\leq\Phi\quad\text{for each}\quad t,s\in[a,b]\quad\text{with } t\neq s; \tag{3.18}$$

(iii) *We have the inequality:*

$$\varphi(t-s)\leq u(t)-u(s)\leq\Phi(t-s)\quad\text{for each}\quad t,s\in[a,b]\quad\text{with } t>s. \tag{3.19}$$

Following [10], we can introduce the concept:

Definition 1. *The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be $(\varphi,\Phi)-$Lipschitzian on $[a,b]$.*

Notice that in [10], the definition was introduced on utilizing the statement (iii) and only the equivalence (i) $\Leftrightarrow$ (iii) was considered.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of $(\varphi,\Phi)-$Lipschitzian functions.

Proposition 1. *Let $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If*

$$-\infty<\gamma:=\inf_{t\in(a,b)}u'(t),\qquad\sup_{t\in(a,b)}u'(t)=:\Gamma<\infty \tag{3.20}$$

then u is $(\gamma,\Gamma)-$Lipschitzian on $[a,b]$.

The following corollary that provides a perturbed version of Taylor's expansion holds:

Corollary 4. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $g : I \to \mathbb{R}$ is such that the n-th derivative $g^{(n)}$ is (l_n, L_n) $-$Lipschitzian with the constants $L_n > l_n > 0$ on the interval $[m, M]$, then for any $c \in [m, M]$ we have the inequality*

$$\begin{aligned}(3.21)\quad & \Bigg| \langle g(A)x, y\rangle - g(c)\langle x, y\rangle - \sum_{k=1}^{n} \frac{1}{k!} g^{(k)}(c) \left\langle (A - c1_H)^k x, y\right\rangle - \frac{l_n + L_n}{2} \\ & \times \left[\frac{1}{(n+1)!}\left\langle A^{n+1}x, y\right\rangle - \frac{c^{n+1}}{(n+1)!}\langle x, y\rangle \right. \\ & \left. - \sum_{k=1}^{n} \frac{c^{n-k+1}}{k!(n-k+1)!}\left\langle (A - c1_H)^k x, y\right\rangle\right]\Bigg| \\ & \leq \frac{1}{2(n+1)!}(L_n - l_n) \\ & \times \left[(c-m)^{n+1}\bigvee_m^c\left(\langle E_{(\cdot)}x, y\rangle\right) + (M-c)^{n+1}\bigvee_c^M\left(\langle E_{(\cdot)}x, y\rangle\right)\right] \\ & \leq \frac{1}{2(n+1)!}(L_n - l_n)\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1}\bigvee_m^M\left(\langle E_{(\cdot)}x, y\rangle\right) \\ & \leq \frac{1}{2(n+1)!}(L_n - l_n)\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1}\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

Proof. Consider the function $f : I \to \mathbb{R}$ defined by

$$f(t) := g(t) - \frac{1}{(n+1)!}\frac{L_n + l_n}{2} \cdot t^{n+1}.$$

Observe that

$$f^{(k)}(t) := g^{(k)}(t) - \frac{1}{(n-k+1)!}\frac{L_n + l_n}{2} \cdot t^{n-k+1}$$

for any $k = 0, ..., n$.

Since $g^{(n)}$ is (l_n, L_n) $-$Lipschitzian it follows that

$$f^{(n)}(t) := g^{(n)}(t) - \frac{L_n + l_n}{2} \cdot t$$

is $\frac{1}{2}(L_n - l_n)$-Lipschitzian and applying Theorem 7 for the function f, we deduce after required calculations the desired result (3.1). □

Remark 2. *In particular, we can state from (3.21) the following inequalities*

$$\begin{aligned}(3.22)\quad &\Bigg|\langle g(A)x,y\rangle - g(m)\langle x,y\rangle - \sum_{k=1}^{n}\frac{1}{k!}g^{(k)}(m)\left\langle (A-m1_H)^k x,y\right\rangle - \frac{l_n+L_n}{2} \\ &\times\Bigg[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{m^{n+1}}{(n+1)!}\langle x,y\rangle \\ &-\sum_{k=1}^{n}\frac{m^{n-k+1}}{k!\,(n-k+1)!}\left\langle (A-m1_H)^k x,y\right\rangle\Bigg]\Bigg| \\ &\le \frac{1}{2\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{2\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\,\|y\|\end{aligned}$$

and

$$\begin{aligned}(3.23)\quad &\Bigg|\langle g(A)x,y\rangle - g(M)\langle x,y\rangle - \sum_{k=1}^{n}\frac{(-1)^k}{k!}g^{(k)}(M)\left\langle (M1_H-A)^k x,y\right\rangle \\ &-\frac{l_n+L_n}{2}\Bigg[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{M^{n+1}}{(n+1)!}\langle x,y\rangle \\ &-\sum_{k=1}^{n}(-1)^k\frac{M^{n-k+1}}{k!\,(n-k+1)!}\left\langle (M1_H-A)^k x,y\right\rangle\Bigg]\Bigg| \\ &\le \frac{1}{2\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{2\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\,\|y\|\end{aligned}$$

and

$$\begin{aligned}(3.24)\quad &\Bigg|\langle g(A)x,y\rangle - g\left(\frac{m+M}{2}\right)\langle x,y\rangle \\ &-\sum_{k=1}^{n}\frac{1}{k!}g^{(k)}\left(\frac{m+M}{2}\right)\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle \\ &-\frac{l_n+L_n}{2}\Bigg[\frac{1}{(n+1)!}\left\langle A^{n+1}x,y\right\rangle - \frac{1}{(n+1)!}\langle x,y\rangle\left(\frac{m+M}{2}\right)^{n+1} \\ &-\sum_{k=1}^{n}\frac{1}{(n-k+1)!k!}\left(\frac{m+M}{2}\right)^{n-k+1}\left\langle\left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle\Bigg]\Bigg| \\ &\le \frac{1}{2^{n+2}\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &\le \frac{1}{2^{n+2}\,(n+1)!}(L_n-l_n)(M-m)^{n+1}\|x\|\,\|y\|\end{aligned}$$

respectively, for any $x,y\in H$.

4. Applications

By utilizing Theorem 6 and 7 for the exponential function, we can state the following result:

Proposition 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c \in [m, M]$ we have the inequality*

$$\left|\left\langle e^{A}x, y\right\rangle - e^{c}\sum_{k=0}^{n}\frac{1}{k!}\left\langle (A - c1_H)^k x, y\right\rangle\right| \tag{4.1}$$
$$\leq \frac{1}{n!}\left[(c-m)^n (e^c - e^m)\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + (M-c)^n\left(e^M - e^c\right)\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$$
$$\leq \frac{1}{n!}\max\left\{(M-c)^n\left(e^M - e^c\right), (c-m)^n(e^c - e^m)\right\}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$$
$$\leq \frac{1}{n!}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n\left(e^M - e^m\right)\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$$
$$\leq \frac{1}{n!}\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^n\left(e^M - e^m\right)\|x\|\,\|y\|$$

and

$$\left|\left\langle e^{A}x, y\right\rangle - e^{c}\sum_{k=0}^{n}\frac{1}{k!}\left\langle (A - c1_H)^k x, y\right\rangle\right| \tag{4.2}$$
$$\leq \frac{1}{(n+1)!}e^M\left[(c-m)^{n+1}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) + (M-c)^{n+1}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$$
$$\leq \frac{1}{(n+1)!}e^M\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$$
$$\leq \frac{1}{(n+1)!}e^M\left(\frac{1}{2}(M-m) + \left|c - \frac{m+M}{2}\right|\right)^{n+1}\|x\|\,\|y\|$$

for any $x, y \in H$.

Remark 3. *We observe that the best inequalities we can get from (4.1) and (4.2) are*

$$\left|\left\langle e^{A}x, y\right\rangle - e^{\frac{m+M}{2}}\sum_{k=0}^{n}\frac{1}{k!}\left\langle \left(A - \frac{m+M}{2}1_H\right)^k x, y\right\rangle\right| \tag{4.3}$$
$$\leq \frac{1}{2^n n!}(M-m)^n\left(e^M - e^m\right)\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$$
$$\leq \frac{1}{2^n n!}(M-m)^n\left(e^M - e^m\right)\|x\|\,\|y\|$$

and

$$\left|\left\langle e^{A}x,y\right\rangle - e^{\frac{m+M}{2}}\sum_{k=0}^{n}\frac{1}{k!}\left\langle\left(A-\frac{m+M}{2}1_H\right)^{k}x,y\right\rangle\right| \tag{4.4}$$
$$\leq \frac{1}{2^{n+1}(n+1)!}e^{M}(M-m)^{n+1}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{2^{n+1}(n+1)!}e^{M}(M-m)^{n+1}\|x\|\|y\|$$

for any $x,y\in H$.

The same Theorems 6 and 7 applied for the logarithmic function produce:

Proposition 3. *Let A be a positive definite operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]\subset(0,\infty)$ and $\{E_\lambda\}_\lambda$ be its spectral family, then for any $c\in[m,M]$ we have the inequalities*

$$\left|\langle \ln Ax,y\rangle - \langle x,y\rangle\ln c - \sum_{k=1}^{n}\frac{(-1)^{k-1}\left\langle (A-c1_H)^{k}x,y\right\rangle}{kc^{k}}\right| \tag{4.5}$$
$$\leq \frac{1}{n}\left[\frac{(c-m)^{n}(c^{n}-m^{n})}{c^{n}m^{n}}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right.$$
$$\left.+\frac{(M-c)^{n}(M^{n}-c^{n})}{M^{m}c^{m}}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\leq \frac{1}{n}\max\left\{\frac{(c-m)^{n}(c^{n}-m^{n})}{c^{n}m^{n}},\frac{(M-c)^{n}(M^{n}-c^{n})}{M^{m}c^{m}}\right\}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{n}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n}\frac{(M^{n}-m^{n})}{M^{m}m^{m}}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{n}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n}\frac{(M^{n}-m^{n})}{M^{m}m^{m}}\|x\|\|y\|$$

and

$$\left|\langle \ln Ax,y\rangle - \langle x,y\rangle\ln c - \sum_{k=1}^{n}\frac{(-1)^{k-1}\left\langle (A-c1_H)^{k}x,y\right\rangle}{kc^{k}}\right| \tag{4.6}$$
$$\leq \frac{1}{(n+1)m^{n+1}}\left[(c-m)^{n+1}\bigvee_{m}^{c}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)+(M-c)^{n+1}\bigvee_{c}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\right]$$
$$\leq \frac{1}{(n+1)m^{n+1}}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\leq \frac{1}{(n+1)m^{n+1}}\left(\frac{1}{2}(M-m)+\left|c-\frac{m+M}{2}\right|\right)^{n+1}\|x\|\|y\|$$

for any $x,y\in H$.

Remark 4. *The best inequalities we can get from (4.5) and (4.6) are*

$$\left|\langle \ln Ax,y\rangle - \langle x,y\rangle \ln\left(\frac{m+M}{2}\right) - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle \left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle}{k\left(\frac{m+M}{2}\right)^k}\right| \tag{4.7}$$

$$\leq \frac{1}{2^n n}(M-m)^n \frac{(M^n-m^n)}{M^m m^m} \bigvee_m^M (\langle E_{(\cdot)}x,y\rangle)$$

$$\leq \frac{1}{2^n n}(M-m)^n \frac{(M^n-m^n)}{M^m m^m} \|x\| \|y\|$$

and

$$\left|\langle \ln Ax,y\rangle - \langle x,y\rangle \ln\left(\frac{m+M}{2}\right) - \sum_{k=1}^{n} \frac{(-1)^{k-1}\left\langle \left(A-\frac{m+M}{2}1_H\right)^k x,y\right\rangle}{k\left(\frac{m+M}{2}\right)^k}\right| \tag{4.8}$$

$$\leq \frac{1}{2^{n+1}(n+1)}\left(\frac{M}{m}-1\right)^{n+1} \bigvee_m^M (\langle E_{(\cdot)}x,y\rangle)$$

$$\leq \frac{1}{2^{n+1}(n+1)}\left(\frac{M}{m}-1\right)^{n+1} \|x\| \|y\|$$

for any $x,y \in H$.

References

[1] S.S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, *Linear and Multilinear Algebra*, Volume **58**, Issue 7, First published 2010, Pages 805 – 814. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[2] S.S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[3] S.S. Dragomir, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, Prerpint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: http://www.staff.vu.edu.au/RGMIA/v11(E).asp].

[4] S.S. Dragomir, Some trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), to appear.

[5] S.S. Dragomir, Some Ostrowski's type vector inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), to appear.

[6] S.S. Dragomir, Comparison between functions of selfadjoint operators in Hilbert spaces and integral means, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), to appear.

[7] S.S. Dragomir, Some generalized trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), to appear.

[8] S.S. Dragomir, Some inequalities for power series of selfadjoint operators in Hilbert spaces via reverses of the Schwarz inequality. *Integral Transforms Spec. Funct.* **20** (2009), no. 9-10, 757–767.

[9] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[10] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.

[11] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, 551–564.

[12] B. Mond and J. Pečarić, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), 405-420.

[13] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.

[14] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207.

Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/dragomir/