# SOME NEW HADAMARD'S TYPE INEQUALITIES FOR CO-ORDINATED $m-$CONVEX AND $(\alpha, m)-$CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some new Hermite-Hadamard's type inequalities of $m-$convex and $(\alpha, m)-$convex functions of 2−variables on the co-ordinates.

## 1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [5]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

In [8], Definition of $m-$convexity was introduced by G.Toader as the following:

**Definition 1.** *The function $f : [0, b] \rightarrow R$, $b > 0$ is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

*for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that $f$ is $m-$concave if $-f$ is $m-$convex.*

Denote by $K_m(b)$ the class of all $m-$convex functions on $[0, b]$ for which $f(0) \leq 0$. Obviously, if we choose $m = 1$, Definition 1 recaptures the concept of standard convex functions on $[0, b]$.

In [6], S.S. Dragomir and G. Toader proved the following Hadamard type inequalities for $m-$convex functions.

**Theorem 1.** *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}, \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.1}$$

Some generalizations of this result can be found in [2],[3].

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**Theorem 2.** *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex differentiable function with $m\in(0,1]$. Then for all $0\leq a<b$ the following inequality holds:*

$$\begin{aligned}\frac{f(mb)}{m}-\frac{b-a}{2}f'(mb) &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{(b-ma)f(b)-(a-mb)f(a)}{2(b-a)}.\end{aligned}\tag{1.2}$$

Also, in [5], S.S. Dragomir proved the following Hadamard type inequalities for $m-$convex functions.

**Theorem 3.** *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$ and $0\leq a<b$.If $f\in L_1[a,b]$, then one has the inequality:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx.\tag{1.3}$$

In [7] , Definition of $(\alpha,m)-$ convexity was introduced by V. G. Miheşan as the following :

**Definition 2.** *The function $f:[0,b]\to\mathbb{R}$, $b>0$ , is said to be $(\alpha,m)-$ convex, where $(\alpha,m)\in[0,1]^2$, if we have*

$$f(tx+m(1-t)y)\leq t^\alpha f(x)+m(1-t^\alpha)f(y)$$

*for all $x,y\in[0,b]$ and $t\in[0,1]$.*

Denote by $K_m^\alpha(b)$ the class of all $(\alpha,m)-$convex functions on $[0,b]$ for which $f(0)\leq0$. If we take $(\alpha,m)=(1,m)$, it can be easily seen that $(\alpha,m)-$ convexity reduces to $m-$ convexity and for $(\alpha,m)=(1,1)$ , $(\alpha,m)-$ convexity reduces to the concept of usual convexity defined on $[0,b]$ , $b>0$.

In [9], E. Set, M. Sardari, M.E. Ozdemir and J. Rooin proved the following Hadamard type inequalities for $(\alpha,m)-$convex functions.

**Theorem 4.** *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]\cap L_1\left[\frac{a}{m},\frac{b}{m}\right]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b\frac{f(x)+m(2^\alpha-1)f\left(\frac{x}{m}\right)}{2^\alpha}dx.\tag{1.4}$$

**Theorem 5.** *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\min\left\{\frac{f(a)+m\alpha f\left(\frac{b}{m}\right)}{\alpha+1},\frac{f(b)+m\alpha f\left(\frac{a}{m}\right)}{\alpha+1}\right\}.\tag{1.5}$$

**Theorem 6.** *Let $f:[0,\infty)\to\mathbb{R}$ be an $(\alpha,m)-$convex function with $(\alpha,m)\in(0,1]^2$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{1}{2}\left[\frac{f(a)+f(b)+m\alpha f\left(\frac{a}{m}\right)+m\alpha f\left(\frac{b}{m}\right)}{\alpha+1}\right].\tag{1.6}$$

Let us now consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on $\Delta$ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t) f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$. A function $f : \Delta \rightarrow \mathbb{R}$ is said to be on the co-ordinates on $\Delta$ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$ (see [5, p. 317]).

Also, in [4], Dragomir proved the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

**Theorem 7.** *Suppose that $f : \Delta \rightarrow \mathbb{R}$ is co-ordinated convex on $\Delta$.Then one has the inequalities:*

$$\begin{aligned}
(1.7)\qquad & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)\,dxdy \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x, c)\,dx + \frac{1}{b-a}\int_a^b f(x, d)\,dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a, y)\,dy + \frac{1}{d-c}\int_c^d f(b, y)\,dy\right] \\
\leq\ & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\end{aligned}$$

*The above inequalities are sharp.*

For co-ordinated s-convex functions, another version of this result can be found in [1].

The main purpose of this paper is to establish new Hadamard-type inequalities of $m-$convex and $(\alpha, m)-$convex functions of 2-variables on the co-ordinates.

## 2. Inequalities for co-ordinated $m-$convex functions

Firstly, we can define the $m$-convex function on co-ordinates, as the following:

**Definition 3.** *Consider the bidimensional interval $\Delta := [0, b] \times [0, d]$ in $[0, \infty)^2$. The mapping $f : \Delta \rightarrow \mathbb{R}$ is $m-$convex on $\Delta$ if*

$$f(tx + (1-t)z, ty + m(1-t)w) \leq tf(x, y) + m(1-t) f(z, w)$$

*holds for all $(x, y), (z, w) \in \Delta$ with $t \in [0, 1]$ and for some fixed $m \in [0, 1]$.*

A function $f : \Delta \rightarrow \mathbb{R}$ is $m-$convex on $\Delta$ is called co-ordinated $m-$convex on $\Delta$ if the partial mappings

$$f_y : [0, b] \rightarrow \mathbb{R}, \quad f_y(u) = f(u, y)$$

and

$$f_x : [0, d] \rightarrow \mathbb{R}, \quad f_x(v) = f(x, v)$$

are $m-$convex for all $y\in[0,d]$ and $x\in[0,b]$ with some fixed $m\in[0,1]$.

**Lemma 1.** *Every $m-$convex mapping $f:\Delta\to[0,\infty)$ is $m-$convex on the co-ordinates.*

*Proof.* Suppose that $f:\Delta=[0,b]\times[0,d]\to[0,\infty)$ is $m-$convex on $\Delta$. Consider the function

$$f_x:[0,d]\to[0,\infty),\quad f_x(v)=f(x,v),\qquad (x\in[0,b]).$$

Then for $t\in[0,1]$, $m\in[0,1]$ and $v_1,v_2\in[0,d]$, we have

$$\begin{aligned} f_x(tv_1+m(1-t)v_2) &= f(x,tv_1+m(1-t)v_2)\\ &= f(tx+(1-t)x,tv_1+m(1-t)v_2)\\ &\le tf(x,v_1)+m(1-t)f(x,v_2)\\ &= tf_x(v_1)+m(1-t)f_x(v_2). \end{aligned}$$

Therefore, $f_x(v)=f(x,v)$ is $m-$convex on $[0,d]$. The fact that $f_y:[0,b]\to[0,\infty)$, $f_y(u)=f(u,y)$ is also $m-$convex on $[0,b]$ for all $y\in[0,d]$ goes likewise and we shall omit the details. □

**Theorem 8.** *Suppose that $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ is $m-$convex function on the co-ordinates on $\Delta$. If $0\le a<b<\infty$ and $0\le c<d<\infty$ with $m\in(0,1]$, then one has the inequality:*

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \\ &\le \frac{1}{4(b-a)}\min\{v_1,v_2\}+\frac{1}{4(d-c)}\min\{v_3,v_4\} \end{aligned}\tag{2.1}$$

*where*

$$\begin{aligned} v_1 &= \int_a^b f(x,c)\,dx+m\int_a^b f\left(x,\frac{d}{m}\right)dx\\ v_2 &= \int_a^b f(x,d)\,dx+m\int_a^b f\left(x,\frac{c}{m}\right)dx\\ v_3 &= \int_c^d f(a,y)\,dy+m\int_c^d f\left(\frac{b}{m},y\right)dy\\ v_4 &= \int_c^d f(b,y)\,dy+m\int_c^d f\left(\frac{a}{m},y\right)dy. \end{aligned}$$

*Proof.* Since $f:\Delta\to\mathbb{R}$ is co-ordinated $m-$convex on $\Delta$ it follows that the mapping $g_x:[0,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is $m-$convex on $[0,d]$ for all $x\in[0,b]$. Then by inequality of (1.1) one has:

$$\frac{1}{d-c}\int_c^d g_x(y)\,dy\le\min\left\{\frac{g_x(c)+mg_x\left(\frac{d}{m}\right)}{2},\frac{g_x(d)+mg_x\left(\frac{c}{m}\right)}{2}\right\}$$

or

$$\frac{1}{d-c}\int_c^d f(x,y)\,dy\le\min\left\{\frac{f(x,c)+mf\left(x,\frac{d}{m}\right)}{2},\frac{f(x,d)+mf\left(x,\frac{c}{m}\right)}{2}\right\}$$

where $0\le c<d<\infty$ and $m\in(0,1]$.

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}(2.2)\quad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \le\ & \min\left\{\frac{1}{2(b-a)}\int_a^b f(x,c)\,dx+\frac{m}{2(b-a)}\int_a^b f\left(x,\frac{d}{m}\right)dx,\right. \\ & \left.\frac{1}{2(b-a)}\int_a^b f(x,d)\,dx+\frac{m}{2(b-a)}\int_a^b f\left(x,\frac{c}{m}\right)dx\right\} \\ =\ & \frac{1}{2(b-a)}\min\left\{\int_a^b f(x,c)\,dx+m\int_a^b f\left(x,\frac{d}{m}\right)dx,\right. \\ & \left.\int_a^b f(x,d)\,dx+m\int_a^b f\left(x,\frac{c}{m}\right)dx\right\}\end{aligned}$$

where $0\le a<b<\infty$.

By a similar argument applied for the mapping $g_y:[0,b]\to[0,\infty)$, $g_y(x)=f(x,y)$ with $0\le a<b<\infty$, we get

$$\begin{aligned}(2.3)\quad & \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \\ \le\ & \frac{1}{2(d-c)}\min\left\{\int_c^d f(a,y)\,dy+m\int_c^d f\left(\frac{b}{m},y\right)dy,\right. \\ & \left.\int_c^d f(b,y)\,dy+m\int_c^d f\left(\frac{a}{m},y\right)dy\right\}.\end{aligned}$$

Summing the inequalities (2.2) and (2.3), we get the inequality (2.1). □

**Corollary 1.** *With the above assumptions and provided that $f$ is differentiable on $\Delta$, we have the inequalities*

$$\begin{aligned}(2.4)\quad & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \le\ & \frac{1}{4(b-a)} \\ & \min\left\{(b-ma)\left[f(b,c)+mf\left(b,\frac{d}{m}\right)\right]-(a-mb)\left[f(a,c)+mf\left(a,\frac{d}{m}\right)\right],\right. \\ & \left.(b-ma)\left[f(b,d)+mf\left(b,\frac{c}{m}\right)\right]-(a-mb)\left[f(a,d)+mf\left(a,\frac{c}{m}\right)\right]\right\},\end{aligned}$$

*and*

$$\begin{aligned}(2.5)\quad & \frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \\ \le\ & \frac{1}{4(d-c)} \\ & \times\min\left\{(d-mc)\left[f(a,d)+mf\left(\frac{b}{m},d\right)\right]-(c-md)\left[f(a,c)+mf\left(\frac{b}{m},c\right)\right],\right. \\ & \left.(d-mc)\left[f(b,d)+mf\left(\frac{a}{m},d\right)\right]-(c-md)\left[f(b,c)+mf\left(\frac{a}{m},c\right)\right]\right\}.\end{aligned}$$

*Proof.* Since $f$ is differentiable on $\Delta$, by the second inequality of (1.2), we have

$$\frac{1}{(b-a)}\int_a^b f(x,c)\,dx \le \frac{(b-ma)\,f(b,c)-(a-mb)\,f(a,c)}{2\,(b-a)}$$

$$\frac{1}{(b-a)}\int_a^b f\left(x,\frac{d}{m}\right)dx \le \frac{(b-ma)\,f(b,\frac{d}{m})-(a-mb)\,f(a,\frac{d}{m})}{2\,(b-a)}$$

$$\frac{1}{(b-a)}\int_a^b f(x,d)\,dx \le \frac{(b-ma)\,f(b,d)-(a-mb)\,f(a,d)}{2\,(b-a)}$$

and

$$\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c}{m}\right)dx \le \frac{(b-ma)\,f(b,\frac{c}{m})-(a-mb)\,f(a,\frac{c}{m})}{2\,(b-a)}.$$

Hence, using the inequality of (2.2), we get the inequality in (2.4).

Analogously, using the inequality of (2.3), we get the inequality in (2.5). The proof is completed. □

**Remark 1.** *Choosing $m=1$ in (2.4) or (2.5), we get relationship between the third and fourth inequalities in (1.7).*

**Theorem 9.** *Suppose that $f:\Delta=[0,b]\times[0,d]\to[0,\infty)$ is $m-$convex function on the co-ordinates on $\Delta$. If $0\le a<b<\infty$ and $0\le c<d<\infty$ , $m\in(0,1]$ with $f_x\in L_1[0,d]$ and $f_y\in L_1[0,b]$ , then one has the inequalities:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\ \le\ & \frac{1}{(b-a)(d-c)}\left[\int_a^b\int_c^d \frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}dydx\right. \\ &\left.+\int_c^d\int_a^b \frac{f(x,y)+mf\left(\frac{x}{m},y\right)}{2}dxdy\right]. \end{aligned} \tag{2.6}$$

*Proof.* Since $f:\Delta\to[0,\infty)$ is co-ordinated $m-$convex on $\Delta$ it follows that the mapping $g_x:[0,d]\to[0,\infty)$ , $g_x(y)=f(x,y)$ is $m-$convex on $[0,d]$ for all $x\in[0,b]$ . Then by inequality of (1.3) one has:

$$g_x\left(\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d\frac{g_x(y)+mg_x\left(\frac{y}{m}\right)}{2}dy$$

or

$$f\left(x,\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d\frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}dy$$

for all $x\in[0,b]$ .

Integrating this inequality on $[a,b]$ , we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\ \le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\frac{f(x,y)+mf\left(x,\frac{y}{m}\right)}{2}dydx. \end{aligned} \tag{2.7}$$

By a similar argument applied for the mapping $g_y : [0,b] \to [0,\infty)$, $g_y(x) = f(x,y)$, we get

$$\begin{aligned} &\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\ \leq\ &\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b \frac{f(x,y)+mf\left(\frac{x}{m},y\right)}{2}dxdy. \end{aligned} \tag{2.8}$$

Summing the inequalities (2.7) and (2.8), we get the inequality in (2.6). □

**Remark 2.** *Choosing $m=1$ in (2.6), we get the second inequality of (1.7).*

## 3. Inequalities for co-ordinated $(\alpha,m)-$convex functions

**Definition 4.** *Consider the bidimensional interval $\Delta := [0,b]\times[0,d]$ in $[0,\infty)^2$. The mapping $f:\Delta\to\mathbb{R}$ is $(\alpha,m)-$convex on $\Delta$ if*

$$f(tx+(1-t)z, ty+m(1-t)w) \leq t^{\alpha}f(x,y)+m(1-t^{\alpha})f(z,w) \tag{3.1}$$

*holds for all $(x,y),(z,w)\in\Delta$ and $(\alpha,m)\in[0,1]^2$ with $t\in[0,1]$.*

A function $f:\Delta\to\mathbb{R}$ is $(\alpha,m)-$convex on $\Delta$ is called co-ordinated $(\alpha,m)-$convex on $\Delta$ if the partial mappings

$$f_y:[0,b]\to\mathbb{R},\quad f_y(u)=f(u,y)$$

and

$$f_x:[0,d]\to\mathbb{R},\quad f_x(v)=f(x,v)$$

are $(\alpha,m)-$convex for all $y\in[0,d]$ and $x\in[0,b]$ with some fixed $(\alpha,m)\in[0,1]^2$.

Note that for $(\alpha,m)=\{(1,m),(1,1)\}$ one obtains the following classes of function : co-ordinated convex and co-ordinated $m-$convex on $\Delta$.

**Lemma 2.** *Every $(\alpha,m)-$convex mapping $f:\Delta\to[0,\infty)$ is $(\alpha,m)-$convex on the co-ordinates.*

*Proof.* Suppose that $f:\Delta\to[0,\infty)$ is $(\alpha,m)-$convex on $\Delta$. Consider the function

$$f_x:[0,d]\to[0,\infty),\quad f_x(v)=f(x,v),\qquad (x\in[0,b]).$$

Then for $t\in[0,1]$, $(\alpha,m)\in[0,1]^2$ and $v_1,v_2\in[0,d]$ one has

$$\begin{aligned} f_x(tv_1+m(1-t)v_2) &= f(x,tv_1+m(1-t)v_2) \\ &= f(tx+(1-t)x,tv_1+m(1-t)v_2) \\ &\leq t^{\alpha}f(x,v_1)+m(1-t^{\alpha})f(x,v_2) \\ &= t^{\alpha}f_x(v_1)+m(1-t^{\alpha})f_x(v_2). \end{aligned}$$

Therefore, $f_x(v)=f(x,v)$ is $(\alpha,m)-$convex on $[0,d]$. The fact that $f_y:[0,b]\to[0,\infty)$, $f_y(u)=f(u,y)$ is also $(\alpha,m)-$convex on $[0,b]$ for all $y\in[0,d]$ goes likewise and we shall omit the details. □

**Theorem 10.** *Suppose that $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ is $(\alpha,m)-$convex function on the co-ordinates on $\Delta$. If $0\leq a<b<\infty$ and $0\leq c<d<\infty$ with $(\alpha,m)\in(0,1]^2$*

*with $f_x \in L_1[0,d]$ and $f_y \in L_1[0,b]$, then the following inequalities hold:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \\ \leq\; &\frac{1}{(d-c)(b-a)} \\ &\times\int_c^d\int_a^b \frac{2f(x,y)+m(2^\alpha-1)\left(f\left(x,\frac{y}{m}\right)+f\left(\frac{x}{m},y\right)\right)}{2^\alpha}dxdy, \end{aligned} \tag{3.2}$$

*and*

$$\begin{aligned} &\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \\ \leq\; &\frac{1}{2(\alpha+1)(b-a)}\min\{w_1,w_2\}+\frac{1}{2(\alpha+1)(d-c)}\min\{w_3,w_4\} \end{aligned} \tag{3.3}$$

*where*

$$\begin{aligned} w_1 &= \int_a^b f(x,c)\,dx+\alpha m\int_a^b f\left(x,\frac{d}{m}\right)dx \\ w_2 &= \int_a^b f(x,d)\,dx+\alpha m\int_a^b f\left(x,\frac{c}{m}\right)dx \\ w_3 &= \int_c^d f(a,y)\,dy+\alpha m\int_c^d f\left(\frac{b}{m},y\right)dy \\ w_4 &= \int_c^d f(b,y)\,dy+\alpha m\int_c^d f\left(\frac{a}{m},y\right)dy. \end{aligned}$$

*Proof.* Since $f:\Delta\to\mathbb{R}$ is co-ordinated $(\alpha,m)-$convex on $\Delta$ it follows that the mapping $g_x:[0,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is $(\alpha,m)-$convex on $[0,d]$ for all $x\in[0,b]$. Then by inequality of (1.4) one has:

$$g_x\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d\frac{g_x(y)+m(2^\alpha-1)g_x\left(\frac{y}{m}\right)}{2^\alpha}dy$$

or

$$f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d\frac{f(x,y)+m(2^\alpha-1)f\left(x,\frac{y}{m}\right)}{2^\alpha}dy$$

where $0\leq c<d<\infty$ and $(\alpha,m)\in(0,1]^2$.

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \\ \leq\; &\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d\frac{f(x,y)+m(2^\alpha-1)f\left(x,\frac{y}{m}\right)}{2^\alpha}dydx \end{aligned} \tag{3.4}$$

where $0\leq a<b<\infty$.

By a similar argument applied for the mapping $g_y:[0,b]\to[0,\infty)$, $g_y(x)=f(x,y)$ with $0\le a<b<\infty$, we get

$$\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \le \frac{1}{(d-c)(b-a)}\int_a^b\int_c^d \frac{f(x,y)+m(2^\alpha-1)f\left(\frac{x}{m},y\right)}{2^\alpha}dydx. \tag{3.5}$$

Summing the inequalities (3.4) and (3.5), we get the inequality (3.2).

The inequality (3.3) can be obtained in a similar way to the proof of Theorem 8 by using (1.5). □

**Remark 3.** *If we take $\alpha=1$, (3.2) and (3.3) reduce to (2.6) and (2.1), respectively.*

**Theorem 11.** *Suppose that $f:\Delta=[0,b]\times[0,d]\to\mathbb{R}$ is $(\alpha,m)-$convex function on the co-ordinates on $\Delta$. If $0\le a<b<\infty$ and $0\le c<d<\infty$ with $(\alpha,m)\in(0,1]^2$ with $f_x\in L_1[0,d]$ and $f_y\in L_1[0,b]$, then the following inequality holds:*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \le\ & \frac{1}{4(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx\\ &+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\\ &\left.+\frac{m\alpha}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy+\frac{m\alpha}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right]\end{aligned} \tag{3.6}$$

*Proof.* Since $f:\Delta\to\mathbb{R}$ is co-ordinated $(\alpha,m)-$convex on $\Delta$ it follows that the mapping $g_x:[0,d]\to\mathbb{R}$, $g_x(y)=f(x,y)$ is $(\alpha,m)-$convex on $[0,d]$ for all $x\in[0,b]$. Then by inequality of (1.6) one has:

$$\frac{1}{d-c}\int_c^d g_x(y)\,dy\le\frac{1}{2}\left[\frac{g_x(c)+g_x(d)+m\alpha\left(g_x\left(\frac{c}{m}\right)+g_x\left(\frac{d}{m}\right)\right)}{\alpha+1}\right]$$

or

$$\frac{1}{d-c}\int_c^d f(x,y)\,dy\le\frac{1}{2}\left[\frac{f(x,c)+f(x,d)+m\alpha\left(f\left(x,\frac{c}{m}\right)+f\left(x,\frac{d}{m}\right)\right)}{\alpha+1}\right]$$

where $0\le c<d<\infty$ and $(\alpha,m)\in(0,1]^2$.

Integrating this inequality on $[a,b]$, we have

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \le\ & \frac{1}{2(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{c}{m}\right)dx+\frac{m\alpha}{b-a}\int_a^b f\left(x,\frac{d}{m}\right)dx\right]\end{aligned} \tag{3.7}$$

where $0 \leq a < b < \infty$.

By a similar argument applied for the mapping $g_y : [0,b] \rightarrow [0,\infty)$, $g_y(x) = f(x,y)$ with $0 \leq a < b < \infty$, we get

$$\begin{aligned}
&\frac{1}{(d-c)(b-a)}\int_c^d\int_a^b f(x,y)\,dxdy \qquad (3.8)\\
&\leq \frac{1}{2(\alpha+1)}\left[\frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right.\\
&\left. + \frac{m\alpha}{d-c}\int_c^d f\left(\frac{a}{m},y\right)dy + \frac{m\alpha}{d-c}\int_c^d f\left(\frac{b}{m},y\right)dy\right].
\end{aligned}$$

Summing the inequalities (3.7) and (3.8), we get the inequality (3.6). □

**Corollary 2.** *Choosing $m = 1$ in Theorem 11, we get the following inequality*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\leq \frac{1}{4(\alpha+1)}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&+ \frac{\alpha}{b-a}\int_a^b f(x,c)\,dx + \frac{\alpha}{b-a}\int_a^b f(x,d)\,dx\\
&+ \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\\
&\left. + \frac{\alpha}{d-c}\int_c^d f(a,y)\,dy + \frac{\alpha}{d-c}\int_c^d f(b,y)\,dy\right].
\end{aligned}$$

**Remark 4.** *Choosing $(\alpha, m) = (1,1)$ in (3.6), we get the third inequality of (1.7).*

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