INEQUALITIES OF HERMITE-HADAMARD'S TYPE FOR FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE $m-$CONVEX

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Abstract. In this paper, we establish several inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are $m-$convex. Some applications for special means of real numbers are also provided.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality(see, [8]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

In [4], G. Toader considered the class of $m-$convex functions: another intermediate between the usual convexity and starshaped convexity.

Definition 1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

Obviously, for $m=1$ Definition 1 recaptures the concept of standard convex functions on $[a,b]$, and for $m=0$ the concept starshaped functions.

For recent results and generalizations concerning convex functions, see the references of [1],[2],[3] and [5].

In [6], U.S. Kırmacı gave the following lemma. Also, in [7], U.S. Kırmacı and M.E. Özdemir obtained the following inequality for differeftiable mappings which are connected with Hermite-Hadamard's inequality and they used this lemma to prove it.

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Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\\ =\ & (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)dt\right].\end{aligned}$$

Theorem 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{\left(3^{1-\frac{1}{q}}\right)}{8}(b-a)\left(|f'(a)| + |f'(b)|\right).$$

In this article, using functions whose derivatives absolute values are $m-$convex, we obtained new inequalities releted to the left side of Hermite-Hadamard inequality. Finally, we gave some applications for special means of real numbers.

2. Main Results

We start with the following theorem:

Theorem 2. *Let $f : I^\circ \subset [0, b^*] \to \mathbb{R}$, $b^* > 0$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is $m-$convex on $[a, b]$ and $m \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{b-a}{4}\left[L_q\left(|f'(a)|, \frac{|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right) + L_q\left(\frac{|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}, m\left|f'\left(\frac{b}{m}\right)\right|\right)\right]\end{aligned}$$

where $|f'(a)| \neq m\left|f'\left(\frac{b}{m}\right)\right|$, $\frac{b}{m} < b^*$ and L_q is a $q-$logarithmic mean of positive real numbers.

Proof. From Lemma 1, using Hölder integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|\\ \le\ & (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt + \int_{\frac{1}{2}}^1 |1-t|\left|f'(ta+(1-t)b)\right|dt\right]\\ \le\ & (b-a)\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\ & + (b-a)\left[\left(\int_{\frac{1}{2}}^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $|f'|$ is $m-$convex on $[a, b]$, we know that for $t \in [0, 1]$

$$\left|f'(ta+(1-t)b)\right| = \left|f'\left(ta + m(1-t)\frac{b}{m}\right)\right| \le t\left|f'(a)\right| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|,$$

hence

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(t\left|f'(a)\right| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)^q dt\right)^{\frac{1}{q}} \\
&+(b-a)\left(\int_{\frac{1}{2}}^1 (1-t)^p\, dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(t\left|f'(a)\right| + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)^q dt\right)^{\frac{1}{q}} \\
=\ & (b-a)\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{2}\frac{\left(\frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)^{q+1} - \left(m\left|f'\left(\frac{b}{m}\right)\right|\right)^{q+1}}{(q+1)\left(\frac{\left|f'(a)\right| - m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)}\right)^{\frac{1}{q}} \\
&+(b-a)\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{2}\frac{\left(\left|f'(a)\right|\right)^{q+1} - \left(\frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)^{q+1}}{(q+1)\left(\frac{\left|f'(a)\right| - m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)}\right)^{\frac{1}{q}} \\
=\ & (b-a)\frac{1}{(p+1)^{\frac{1}{p}}}\frac{1}{2^{\frac{p+1}{p}}}\frac{1}{2^{\frac{1}{q}}} \\
&\times\left[L_q\left(\frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}, m\left|f'\left(\frac{b}{m}\right)\right|\right)\right. \\
&\left.+L_q\left(\left|f'(a)\right|, \frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right].
\end{aligned}$$

Since $\frac{1}{(p+1)^{\frac{1}{p}}} < 1$ if $p > 1$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)}{4}\left[L_q\left(\frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}, m\left|f'\left(\frac{b}{m}\right)\right|\right)\right. \\
&\left.+L_q\left(\left|f'(a)\right|, \frac{\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which completes the proof. □

Theorem 3. *Let $f : I^{\circ} \subset [0, b^{*}] \to \mathbb{R}$, $b^{*} > 0$, be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $|f'|^{q}$ is $m-$convex on $[a, b]$, $q > 1$ and $m \in (0, 1]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_{a}^{b} f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{4}\left\{\left(|f'(a)|^{q} + 3m\left|f'\left(\frac{b}{m}\right)\right|^{q}\right)^{\frac{1}{q}} + \left(3\,|f'(a)|^{q} + m\left|f'\left(\frac{b}{m}\right)\right|^{q}\right)^{\frac{1}{q}}\right\}$$

where $\frac{b}{m} < b^{*}$.

Proof. From Lemma 1, using Hölder integral inequality, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_{a}^{b} f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & (b-a)\left[\int_{0}^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|\,dt + \int_{0}^{\frac{1}{2}} |1-t|\,|f'(ta+(1-t)b)|\,dt\right] \\ \leq\ & (b-a)\left(\int_{0}^{\frac{1}{2}} t^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}} |f'(ta+(1-t)b)|^{q}\,dt\right)^{\frac{1}{q}} \\ & + (b-a)\left(\int_{\frac{1}{2}}^{1} (1-t)^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} |f'(ta+(1-t)b)|^{q}\,dt\right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f'|^{q}$ is $m-$convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(ta+(1-t)b)|^{q} = \left|f'\left(ta + m(1-t)\frac{b}{m}\right)\right|^{q} \leq t\,|f'(a)|^{q} + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^{q},$$

hence

$$\begin{aligned} &\left|\frac{1}{b-a}\int_{a}^{b} f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & (b-a)\left(\int_{0}^{\frac{1}{2}} t^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left(t\,|f'(a)|^{q} + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^{q}\right)dt\right)^{\frac{1}{q}} \\ & + (b-a)\left(\int_{\frac{1}{2}}^{1} (1-t)^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(t\,|f'(a)|^{q} + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^{q}\right)dt\right)^{\frac{1}{q}} \\ =\ & (b-a)\left(\frac{1}{2^{p+1}\,(p+1)}\right)^{\frac{1}{p}} \\ & \times\left[\left(\frac{|f'(a)|^{q}}{8} + \frac{3m\,|f'\left(\frac{b}{m}\right)|^{q}}{8}\right)^{\frac{1}{q}} + \left(\frac{3\,|f'(a)|^{q}}{8} + \frac{m\,|f'\left(\frac{b}{m}\right)|^{q}}{8}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Since $\left(\frac{1}{p+1}\right)^{\frac{1}{p}} < 1$ and $\frac{1}{4^{1/q}} < 1$ if $p > 1$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{4}\left\{\left(|f'(a)|^q + 3m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(3|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which completes the proof. □

Corollary 1. *In theorem 3, if $m = 1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left\{\left(|f'(a)|^q + 3|f'(b)|^q\right)^{\frac{1}{q}} + \left(3|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Theorem 4. *Let $f : I^\circ \subset [0, b^*] \to \mathbb{R}$, $b^* > 0$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is $m-$convex on $[a, b]$ and $m \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{b-a}{8}\left\{\left[\frac{1}{6} + L_{2q}^{4q^2}\left(m\left|f'\left(\frac{b}{m}\right)\right|, \frac{|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right. \\ & \left. + \left[\frac{1}{6} + L_{2q}^{4q^2}\left(|f'(a)|, \frac{|f'(a)| + m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $|f'(a)| \neq m\left|f'\left(\frac{b}{m}\right)\right|$, $\frac{b}{m} < b^*$ and L_q is a $q-$logarithmic mean of positive real numbers.

Proof. From Lemma 1, using well-known power mean inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq\ & (b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta + (1-t)b)\right|dt + \int_0^{\frac{1}{2}} |1-t|\left|f'(ta + (1-t)b)\right|dt\right] \\ \leq\ & (b-a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\left(|f'(ta + (1-t)b)|\right)^q dt\right)^{\frac{1}{q}}\right] \\ & + (b-a)\left[\left(\int_{\frac{1}{2}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)\left(|f'(ta + (1-t)b)|\right)^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Since $|f'|$ is $m-$convex, using the elementary inequality $cd \leq \frac{1}{2}\left(c^2+d^2\right)$, $(c,d\in\mathbb{R})$, we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} t\left(|f'(ta+(1-t)b)|\right)^q dt \\
\leq\ & \int_0^{\frac{1}{2}} t\left(t\,|f'(a)|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)^q dt \\
\leq\ & \frac{1}{2}\int_0^{\frac{1}{2}} t^2 dt+\frac{1}{2}\int_0^{\frac{1}{2}}\left(t\,|f'(a)|+m(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)^{2q} dt \\
=\ & \frac{1}{48}+\frac{1}{2}\left[\frac{1}{2}\frac{\left(m\left|f'\left(\frac{b}{m}\right)\right|\right)^{2q+1}-\left(\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)^{2q+1}}{(2q+1)\left(\frac{m\left|f'\left(\frac{b}{m}\right)\right|-|f'(a)|}{2}\right)}\right]^{2q} \\
=\ & \frac{1}{48}+\frac{1}{2^{2q+1}}L_{2q}^{4q^2}\left(m\left|f'\left(\frac{b}{m}\right)\right|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)
\end{aligned}$$

and analogously

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 (1-t)\left(|f'(ta+(1-t)b)|\right)^q dt \\
\leq\ & \frac{1}{48}+\frac{1}{2^{2q+1}}L_{2q}^{4q^2}\left(|f'(a)|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)\frac{1}{8^{1-\frac{1}{q}}}\left\{\left[\frac{1}{48}+\frac{1}{2^{2q+1}}L_{2q}^{4q^2}\left(m\left|f'\left(\frac{b}{m}\right)\right|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right. \\
& \left.+\left[\frac{1}{48}+\frac{1}{2^{2q+1}}L_{2q}^{4q^2}\left(|f'(a)|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{8}\left\{\left[\frac{1}{6}+\frac{1}{2^{2q-2}}L_{2q}^{4q^2}\left(m\left|f'\left(\frac{b}{m}\right)\right|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right. \\
& \left.+\left[\frac{1}{6}+\frac{1}{2^{2q-2}}L_{2q}^{4q^2}\left(|f'(a)|,\frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\frac{1}{2^{2q-2}} \leq 1$ if $q \geq 1$, we obtain

$$\begin{aligned}
&\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{8}\left\{\left[\frac{1}{6}+L_{2q}^{4q^2}\left(m\left|f'\left(\frac{b}{m}\right)\right|, \frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\frac{1}{6}+L_{2q}^{4q^2}\left(|f'(a)|, \frac{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|}{2}\right)\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof. □

Corollary 2. *In theorem 4, if $m = 1$, we have*

$$\begin{aligned}
&\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{8}\left\{\left[\frac{1}{6}+L_{2q}^{4q^2}\left(|f'(b)|, \frac{|f'(a)|+|f'(b)|}{2}\right)\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\frac{1}{6}+L_{2q}^{4q^2}\left(|f'(b)|, \frac{|f'(a)|+|f'(b)|}{2}\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 5. *Let $f : I^\circ \subset [0, b^*] \to \mathbb{R}$, $b^* > 0$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is $m-$convex on $[a, b]$, $q > 1$ and $m \in (0, 1]$, then the following inequality holds:*

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(3^{1-\frac{1}{q}}\right)\left(|f'(a)| + m^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right)$$

where $\frac{b}{m} < b^*$.

Proof. From Lemma 1, using well-known power mean inequality, we have

$$\begin{aligned}
&\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)\left[\int_0^{\frac{1}{2}} t\,|f'(ta+(1-t)b)|\,dt + \int_0^{\frac{1}{2}} |1-t|\,|f'(ta+(1-t)b)|\,dt\right] \\
\leq\ & (b-a)\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t\left(|f'(ta+(1-t)b)|\right)^q dt\right)^{\frac{1}{q}}\right] \\
&+ (b-a)\left[\left(\int_{\frac{1}{2}}^1 (1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)\left(|f'(ta+(1-t)b)|\right)^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'|^q$ is $m-$convex, we have

$$\begin{aligned}
(2.1)\qquad & \int_0^{\frac{1}{2}} t\left(|f'(ta+(1-t)b)|\right)^q dt \\
\le & \int_0^{\frac{1}{2}} t\left(t|f'(a)|^q + m(1-t)\left|f'\left(\frac{b}{m}\right)\right|^q\right) dt \\
= & \int_0^{\frac{1}{2}} \left(t^2|f'(a)|^q + m(t-t^2)\left|f'\left(\frac{b}{m}\right)\right|^q\right) dt \\
= & \frac{|f'(a)|^q}{24} + \frac{m\left|f'\left(\frac{b}{m}\right)\right|^q}{12}.
\end{aligned}$$

Similarly, we have

$$(2.2)\qquad \int_{\frac{1}{2}}^1 (1-t)\left(|f'(ta+(1-t)b)|\right)^q dt \le \frac{|f'(a)|^q}{12} + \frac{m\left|f'\left(\frac{b}{m}\right)\right|^q}{24}.$$

From (2.1) and (2.2), we obtain

$$\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
& \le (b-a)\frac{1}{8^{1-\frac{1}{q}}}\left\{\left[\frac{|f'(a)|^q + 2m\left|f'\left(\frac{b}{m}\right)\right|^q}{24}\right]^{\frac{1}{q}} + \left[\frac{2|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q}{24}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Let $a_1 = |f'(a)|^q$, $b_1 = 2m\left|f'\left(\frac{b}{m}\right)\right|^q$, $a_2 = 2|f'(a)|^q$ and $b_2 = m\left|f'\left(\frac{b}{m}\right)\right|^q$. Here $0 < \frac{1}{q} < 1$, for $q > 1$. Using the fact that

$$\sum_{k=1}^n (a_k + b_k)^s \le \sum_{k=1}^n (a_k)^s + \sum_{k=1}^n (b_k)^s$$

for $(0 \le s < 1)$, $a_1, a_2, ..., a_n \ge 0$, $b_1, b_2, ..., b_n \ge 0$, we obtain

$$\begin{aligned}
& \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\le & (b-a)\frac{1}{8^{1-\frac{1}{q}}}\frac{3}{24^{\frac{1}{q}}}\left(|f'(a)| + m^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right) \\
\le & \frac{b-a}{8}\left(3^{1-\frac{1}{q}}\right)\left(|f'(a)| + m^{\frac{1}{q}}\left|f'\left(\frac{b}{m}\right)\right|\right)
\end{aligned}$$

which completes the proof. □

Remark 1. *In Theorem 5, if $m = 1$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{\left(3^{1-\frac{1}{q}}\right)}{8}(b-a)\left(|f'(a)| + |f'(b)|\right)$$

which the inequality of Theorem 1.

3. Applications to Some Special Means

We now consider the applications of Corollary 1 and Theorem 5 to the following special means:

(a) The arithmetic mean: $A = A(a,b) := (a+b)/2, \;\; a,b \geq 0,$

(b) The Identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a \neq b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a = b \end{cases}, \quad a,b > 0,$$

(c) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R}\diagdown\{-1,0\}\,; \; a,b > 0.$$

The following propositions hold:

Proposition 1. *Let $a,b \in \mathbb{R}_+$, $a < b$ and $\frac{1}{q} \in \mathbb{N}$. Then, for all $q > 1$*

$$\left|L^{q/q+1}_{q+1/q,}(a,b) - A^{q+1/q}(a,b)\right| \leq (b-a)\,A\left((a+3b)^{\frac{1}{q}}, (3a+b)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from Corollary 1 applied for $f(x) = x^{\frac{1}{q}+1}, x \in \mathbb{R}_+$, $\frac{1}{q} \in \mathbb{N}$ and $q > 1$. Also, for $q > 1$, $\frac{1}{q}+1 < 2$ used in the proof. □

Proposition 2. *Let $a,b \in \mathbb{R}_+$, $a < b$, $n \in \mathbb{Z}$ and $n \geq 2$ with $m \in (0,1]$. Then, for all $q > 1$*

$$|L^n_{n,}(a,b) - A^n(a,b)| \leq 2n\frac{\left(3^{1-\frac{1}{q}}\right)}{8}(b-a)\,A\left(a^{n-1}, m^{\frac{1}{q}}\left(\frac{b}{m}\right)^{n-1}\right).$$

Proof. The assertion follows by Theorem 5 on choosing $f : [0,\infty) \to [0,\infty)$, $f(x) = x^n$, $n \in \mathbb{Z}$ and $n \geq 2$ which is $m-$convex on $[0,\infty)$. □

Proposition 3. *Let $a,b \in [0,\infty)$ and $a < b$. Then, for all $q > 1$ we have*

$$\begin{aligned} & \ln\left[\frac{A(a,b)+1}{I(a+1,b+1)}\right] \\ \leq \;& \frac{b-a}{4}\left\{\frac{\left[(b+1)^q + 3(a+1)^q\right]^{1/q} + \left[3(b+1)^q + (a+1)^q\right]^{1/q}}{(b+1)(a+1)}\right\}. \end{aligned}$$

Proof. The proof follows by corollary 1 on choosing $f : [0,\infty) \to (-\infty,0]$, $f(x) = -\ln(x+1)$ which is $m-$convex on $[0,\infty)$, and we omit the details. □

Proposition 4. *With the above assumption for all $q > 1$ and $m \in (0,1]$ we have*

$$\ln\left[\frac{A(a,b)+1}{I(a+1,b+1)}\right] \leq \frac{b-a}{8}\left(3^{1-\frac{1}{q}}\right)\left(\frac{1}{a+1} + \frac{m^{1+\frac{1}{q}}}{b+m}\right).$$

Proof. The proof follows by Theorem 5 for the same function $f : [0,\infty) \to (-\infty,0]$, $f(x) = -\ln(x+1)$ and the details are omitted. □

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