A note on a bound of the combination of arithmetic and harmonic means for the Seiffert's mean

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Abstract

It is shown that the main result of a recent paper by S. Wang and Y. Chu [5] is due in fact to the author [4], [2].

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1 Introduction

Let $a, b > 0$ and $A = A(a,b) = \dfrac{a+b}{2}$, $G = G(a,b) = \sqrt{ab}$, $H = H(a,b) = \dfrac{2ab}{a+b}$ be the classical means representing the arithmetic, geometric and harmonic means of a and b.

Further, let $P = P(a,b) = \dfrac{a-b}{4\operatorname{arctg}\left(\sqrt{\dfrac{a}{b}}\right) - \pi}$, $a \neq b$; $P(a,a) = a$ be the Seiffert mean of a and b. For references of this mean, see the Bibliography of papers [4], [2], [5].

In paper [4] the author remarked that P can be written as the common limit of a pair of sequences (a_n) and (b_n), defined recurrently by

$$a_0 = \sqrt{ab},\ b_0 = \frac{a+b}{2},\ a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_{n+1} \cdot b_n} \quad (n \geq 0).$$

Since this algorithm is due to Pfaff (see e.g. [1]), the author suggested the use of letter ,,P" for this mean.

By using these sequences, the author proved in [4] that

$$\sqrt[3]{b_n^2 a_n} < P < \frac{a_n + 2b_n}{3} \text{ for all } \geq 0. \tag{1}$$

Particularly, the left side of (1) for $n = 0$ gives

$$A^{\frac{2}{3}} \cdot G^{\frac{1}{3}} < P, \tag{2}$$

which is the left side of inequality (20) in [4].

For $n = 1$ we get a better lower bound, namely (see (23) in [4])

$$\left(\frac{A+G}{2}\right)^{\frac{2}{3}} \cdot A^{\frac{1}{3}} < P. \tag{3}$$

We note that from (1) we can deduce better-and-better results for increasing values of n.

2 Main results

In paper [5] it is shown that

$$A^{\frac{5}{6}} \cdot H^{\frac{1}{6}} < P, \tag{4}$$

which in fact may be considered the main result of the paper.

As $H = \dfrac{G^2}{A}$, it is easy to see that $A^{\frac{5}{6}} \cdot H^{\frac{1}{6}} = A^{\frac{2}{3}} \cdot G^{\frac{1}{3}}$, i.e. relation (4) coincides in fact with (2). The complicated method used by the authors in [5] should be compared to the natural sequential method from [4].

The authors prove also that $\dfrac{5}{6}$ is the best value of k in

$$A^k \cdot H^{1-k} < P, \tag{5}$$

but this fact has been remarked also in [2], where a generalization of the method from [4] to the general Schwab–Borchardt mean was deduced. In fact one may consider instead the Seiffert mean P, the more general mean $SB(a, b)$, representing the ,,Schwab–Borchardt" mean (see also [3]).

References

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