THE QUASILINEARITY OF SOME FUNCTIONALS ASSOCIATED WITH THE RIEMANN-STIELTJES INTEGRAL

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Abstract. The superadditivity and subadditivity of some functionals associated with the Riemann-Stieltjes integral are established. Applications in connection to Ostrowski's and the Generalised Trapezoidal inequalities and for special means are provided.

1. Introduction

In theory of the *Riemann-Stieltjes integral* for scalar functions, it is well known that if $f : [a, b] \to \mathbb{R}$ $(\mathbb{C})$ is continuous and $u : [a, b] \to \mathbb{R}$ $(\mathbb{C})$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ exists and the following sharp inequality holds

$$\left| \int_a^b f(t)\, du(t) \right| \leq \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u), \tag{1.1}$$

where $\bigvee_a^b (u)$ denotes the *total variation* of u on $[a, b]$, we recall that

$$\bigvee_a^b (u) = \sup \left\{ \sum_{i=0}^{n-1} |u(x_{i+1}) - u(x_i)|,\ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \right\}.$$

The inequality (1.1) plays an important role in obtaining various sharp bounds for the approximation error of the Riemann-Stieltjes integral by simpler quantities such as:

$$f(x)[u(b) - u(a)], \qquad \text{(see [5], [6], [1])}$$
$$f(b)[u(b) - u(x)] + f(a)[u(x) - u(a)] \qquad \text{(see [7], [1])}$$

and

$$\frac{1}{b-a}[u(b) - u(a)] \int_a^b f(t)\, dt \qquad \text{(see [8], [9])},$$

where $x \in [a, b]$.

Following the recent paper [2], for a continuous function $f : [a, b] \to \mathbb{R}$ and a function of bounded variation $u : [a, b] \to \mathbb{R}$ we define the following functional

$$\Psi(f, u; [a, b]) := \max_{t\in[a,b]} |f(t)| \cdot \bigvee_a^b (u) - \left| \int_a^b f(t)\, du(t) \right|. \tag{1.2}$$

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Due to the properties of the Riemann-Stieltjes integral, the functional Ψ is well defined and nonnegative.

The following properties of this functional as a function of interval hold [2]:

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $c\in(a,b)$ we have*

$$(0\leq)\,\Psi(f,u;[a,c])+\Psi(f,u;[c,b])\leq\Psi(f,u;[a,b]), \tag{1.3}$$

i.e., $\Psi(f,u;\cdot)$ is superadditive as a function of an interval.

If $[c,d]\subseteq[a,b]$, then

$$(0\leq)\,\Psi(f,u;[c,d])\leq\Psi(f,u;[a,b]), \tag{1.4}$$

i.e., $\Psi(f,u;\cdot)$ is monotonic nondecreasing as a function of an interval.

In the same paper the following functional has been considered as well:

$$\Phi(f,u;[a,b]):=\left[\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\bigvee_a^b(u)-\left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^{(b-a)},$$

which is well defined for continuous functions $f:[a,b]\to\mathbb{R}$ and functions of bounded variation $u:[a,b]\to\mathbb{R}$.

The following result concerning the properties of this functional holds [2]:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $c\in(a,b)$ we have*

$$\Phi(f,u;[a,b])\geq\Phi(f,u;[a,c])\cdot\Phi(f,u;[c,b]), \tag{1.5}$$

i.e., $\Phi(f,u;\cdot)$ is supermultiplicative as a function of interval.

For applications of the above results for the *Trapezoidal and Ostrowski error functionals* as well as applications for special means, see [2].

In this paper we consider other composite functionals that can be naturally associated with the functional $\Psi(f,u;[a,b])$ and study their quasilinearity properties. Some applications in connection to Ostrowski's and the Generalised Trapezoidal inequalities and for special means are provided as well.

2. Some General Results

Consider a continuous function $f:[a,b]\to\mathbb{R}$ and a function of bounded variation $u:[a,b]\to\mathbb{R}$ for which

$$\Psi(f,u;[x,y]):=\max_{t\in[x,y]}|f(t)|\cdot\bigvee_x^y(u)-\left|\int_x^y f(t)\,du(t)\right|\neq 0 \tag{2.1}$$

for any proper subinterval $[x,y]$ of the given interval $[a,b]$.

Define the new functional

$$\begin{aligned}\Upsilon(f,u;[a,b])&:=\frac{b-a}{\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\cdot\bigvee_a^b(u)-\left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|}\\&=\frac{(b-a)^2}{\Psi(f,u;[a,b])}.\end{aligned} \tag{2.2}$$

The following result holds:

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$ such that the condition (2.1) is valid. Then for any $c\in(a,b)$ we have*

$$\Upsilon(f,u;[a,b])\leq\Upsilon(f,u;[a,c])+\Upsilon(f,u;[c,b]),\tag{2.3}$$

i.e., $\Upsilon(f,u;\cdot)$ is subadditive as a function of interval.

Proof. Since, by Theorem 1, the functional $\Psi(f,u;\cdot)$ is superadditive as a function of interval, we have for any $c\in(a,b)$ that

$$\frac{\Psi(f,u;[a,b])}{b-a}\geq\frac{\Psi(f,u;[a,c])+\Psi(f,u;[c,b])}{b-a}\tag{2.4}$$
$$=\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}.$$

Utilising the elementary inequality between the *weighted arithmetic mean* and the *weighted harmonic mean*, i.e.,

$$\frac{\alpha a+\beta b}{\alpha+\beta}\geq\frac{\alpha+\beta}{\frac{\alpha}{a}+\frac{\beta}{b}},\alpha,\beta,a,b>0$$

for the choices $a=\frac{\Psi(f,u;[a,c])}{c-a}$, $b=\frac{\Psi(f,u;[c,b])}{b-c}$, $\alpha=c-a$ and $\beta=b-c$, we have

$$\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}\geq\frac{(c-a)+(b-c)}{\frac{c-a}{\frac{\Psi(f,u;[a,c])}{c-a}}+\frac{b-c}{\frac{\Psi(f,u;[c,b])}{b-c}}}\tag{2.5}$$
$$=\frac{(c-a)+(b-c)}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}}=\frac{b-a}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}}.$$

Combining (2.4) with (2.5) we get

$$\frac{\Psi(f,u;[a,b])}{b-a}\geq\frac{b-a}{\frac{(c-a)^2}{\Psi(f,u;[a,c])}+\frac{(b-c)^2}{\Psi(f,u;[c,b])}},$$

which shows that the functional $\Upsilon(f,u;\cdot)$ is subadditive as a function of interval. □

Further, for $q\in(0,1)$, we consider the following family of functionals

$$\Omega_q(f,u;[a,b]):=(b-a)^{1-q}[\Psi(f,u;[a,b])]^q\tag{2.6}$$
$$=(b-a)\left[\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\cdot\bigvee_a^b(u)-\left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^q.$$

The following result concerning the quasilinearity of the functional $\Omega_q(f,u;\cdot)$ may be stated:

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and $u:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then for any $q\in(0,1)$*

$$(0\leq)\,\Omega_q(f,u;[a,c])+\Omega_q(f,u;[c,b])\leq\Omega_q(f,u;[a,b]),\tag{2.7}$$

for any $c\in(a,b)$, i.e., the functional $\Omega_q(f,u;\cdot)$ is superadditive as a function of interval.

If $[c,d]\subseteq[a,b]$, *then*

$$(0\leq)\,\Omega_q(f,u;[c,d])\leq\Omega_q(f,u;[a,b]), \tag{2.8}$$

i.e., $\Omega_q(f,u;\cdot)$ *is monotonic nondecreasing as a function of interval.*

Proof. We know from the proof of Theorem 4 that

$$\frac{\Psi(f,u;[a,b])}{b-a}\geq\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)} \tag{2.9}$$

for any $c\in(a,b)$.

Taking the power $q\in(0,1)$ in (2.9) we get

$$\left[\frac{\Psi(f,u;[a,b])}{b-a}\right]^q\geq\left[\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}\right]^q \tag{2.10}$$

for any $c\in(a,b)$.

By the concavity of the function $g(t)=t^q, q\in(0,1)$ we also have

$$\begin{aligned}&\left[\frac{(c-a)\frac{\Psi(f,u;[a,c])}{c-a}+(b-c)\frac{\Psi(f,u;[c,b])}{b-c}}{(c-a)+(b-c)}\right]^q \\ &\geq\frac{(c-a)\left[\frac{\Psi(f,u;[a,c])}{c-a}\right]^q+(b-c)\left[\frac{\Psi(f,u;[c,b])}{b-c}\right]^q}{(c-a)+(b-c)} \\ &=\frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q+(b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{(c-a)+(b-c)} \\ &=\frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q+(b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{b-a}\end{aligned} \tag{2.11}$$

for any $c\in(a,b)$.

Combining (2.10) with (2.11) we deduce

$$\frac{[\Psi(f,u;[a,b])]^q}{(b-a)^q}\geq\frac{(c-a)^{1-q}[\Psi(f,u;[a,c])]^q+(b-c)^{1-q}[\Psi(f,u;[c,b])]^q}{b-a}$$

for any $c\in(a,b)$, which shows that the functional $\Omega_q(f,u;\cdot)$ is superadditive as a function of interval.

Now, let $a<c<d<b$. Then by the superadditivity of $\Omega_q(f,u;\cdot)$ we have

$$\Omega_q(f,u;[a,b])-\Omega_q(f,u;[c,d])\geq\Omega_q(f,u;[a,c])+\Omega_q(f,u;[d,b])\geq0,$$

which proves the monotonicity property. □

If $p\geq q\geq0, p\geq1$ we can also consider the mapping depending on two parameters:

$$\begin{aligned}\Lambda_{p,q}(f,u;[a,b])&:=(b-a)^{\frac{p-q}{p}}\Psi^q(f,u;[a,b]) \\ &=(b-a)^{\frac{p-q+pq}{p}}\left[\max_{t\in[a,b]}|f(t)|\cdot\frac{1}{b-a}\bigvee_a^b(u)-\left|\frac{1}{b-a}\int_a^b f(t)\,du(t)\right|\right]^q.\end{aligned} \tag{2.12}$$

We have also the following general result:

Theorem 5. *Assume that* $f:[a,b]\to\mathbb{R}$ *is continuous and* $u:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$. *Then for any* $p\geq q\geq 0, p\geq 1$ *we have that the functional* $\Lambda_{p,q}(f,u;\cdot)$ *defined by (2.13) is superadditive and monotonic nondecreasing as a function of interval.*

Proof. First of all, we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p\geq(\leq)\,\alpha^p+\beta^p \tag{2.13}$$

for any $\alpha,\beta\geq 0$ and $p\geq 1$ $(0<p<1)$.

Indeed, if we consider the function $f_p:[0,\infty)\to\mathbb{R}$, $f_p(t)=(t+1)^p-t^p$ we have $f_p'(t)=p\left[(t+1)^{p-1}-t^{p-1}\right]$. Observe that for $p>1$ and $t>0$ we have that $f_p'(t)>0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t=\frac{\alpha}{\beta}$ $(\beta>0,\alpha\geq 0)$ we have $f_p(t)>f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p-\left(\frac{\alpha}{\beta}\right)^p>1$, i.e., the desired inequality (2.1).

For $p\in(0,1)$ we have that f_p is strictly decreasing on $[0,\infty)$ which proves the second case in (2.13).

Now, let $c\in(a,b)$. Since $\Psi(f,u;\cdot)$ is superadditive as a function of interval, then for any $p\geq 1$ we have, by (2.13) that

$$\begin{aligned}\Psi^p(f,u;[a,b])&\geq[\Psi(f,u;[a,c])+\Psi(f,u;[c,b])]^p\\&\geq\Psi^p(f,u;[a,c])+\Psi^p(f,u;[c,b])\end{aligned} \tag{2.14}$$

which provides that

$$\begin{aligned}\frac{\Psi(f,u;[a,b])}{b-a}&\geq\frac{[\Psi^p(f,u;[a,c])+\Psi^p(f,u;[c,b])]^{1/p}}{(c-a)+(b-c)}\\&=\left(\frac{(c-a)\left[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\right]^p+(b-c)\left[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\right]^p}{(c-a)+(b-c)}\right)^{1/p}(b-a)^{\frac{1}{p}-1}\end{aligned} \tag{2.15}$$

for any $c\in(a,b)$.

Utilising the monotonicity property of *power means*, i.e.,

$$\left(\frac{\alpha x^p+\beta y^p}{\alpha+\beta}\right)^{\frac{1}{p}}\geq\left(\frac{\alpha x^q+\beta y^q}{\alpha+\beta}\right)^{\frac{1}{q}}$$

where $p\geq q\geq 0$, and $\alpha,\beta,x,y\geq 0$ with $\alpha+\beta>0$, then we have

$$\begin{aligned}&\left(\frac{(c-a)\left[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\right]^p+(b-c)\left[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\right]^p}{(c-a)+(b-c)}\right)^{1/p}\\&\geq\left(\frac{(c-a)\left[\frac{\Psi(f,u;[a,c])}{(c-a)^{1/p}}\right]^q+(b-c)\left[\frac{\Psi(f,u;[c,b])}{(b-c)^{1/p}}\right]^q}{(c-a)+(b-c)}\right)^{1/q}\\&=\left(\frac{(c-a)^{1-\frac{q}{p}}\,\Psi^q(f,u;[a,c])+(b-c)^{1-\frac{q}{p}}\,\Psi^q(f,u;[c,b])}{b-a}\right)^{1/q}.\end{aligned} \tag{2.16}$$

By making use of the inequalities (2.15) and (2.16), we get

$$\frac{\Psi(f,u;[a,b])}{b-a} \geq \left(\frac{(c-a)^{1-\frac{q}{p}}\Psi^q(f,u;[a,c])+(b-c)^{1-\frac{q}{p}}\Psi^q(f,u;[c,b])}{b-a}\right)^{1/q}(b-a)^{\frac{1}{p}-1}$$

which is equivalent, by taking the power q, with

$$\begin{aligned}\frac{\Psi^q(f,u;[a,b])}{(b-a)^q} &\geq \left(\frac{(c-a)^{1-\frac{q}{p}}\Psi^q(f,u;[a,c])+(b-c)^{1-\frac{q}{p}}\Psi^q(f,u;[c,b])}{b-a}\right)(b-a)^{\frac{q}{p}-q} \\ &= \left[(c-a)^{1-\frac{q}{p}}\Psi^q(f,u;[a,c])+(b-c)^{1-\frac{q}{p}}\Psi^q(f,u;[c,b])\right](b-a)^{\frac{q}{p}-q-1}.\end{aligned} \tag{2.17}$$

Moreover, if we multiply (2.17) with $(b-a)^{1+q-\frac{q}{p}}$, then we get

$$\Psi^q(f,u;[a,b])(b-a)^{1-\frac{q}{p}} \geq (c-a)^{1-\frac{q}{p}}\Psi^q(f,u;[a,c])+(b-c)^{1-\frac{q}{p}}\Psi^q(f,u;[c,b]) \tag{2.18}$$

for any $c\in(a,b)$, which shows that $\Lambda_{p,q}(f,u;\cdot)$ is superadditive as a function of interval.

The monotonicity follows as above and the proof is complete. □

3. Applications for Ostrowski's Inequality

In [6], the author has proved the following inequality of *Ostrowski's type* for the Riemann-Stieltjes integral

$$\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right| \leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(u), \tag{3.1}$$

for all $x\in[a,b]$, where $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type, i.e.

$$|f(x)-f(y)|\leq H|x-y|^r$$

for any $x,y\in[a,b]$, where $r\in(0,1]$ and $H>0$ are given and u is of bounded variation on $[a,b]$.

Now, we can define the following functional:

$$\theta(f,u,x)(a,b):=H\max_{t\in[a,b]}|t-x|^r\bigvee_a^b(u) - \left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right| \tag{3.2}$$

where f,u,x,a,b,r and H are as above.

We observe that, when x is in the interior of $[a,b]$ then

$$\max_{t\in[a,b]}|t-x|^{r}=\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{r}$$

which provides a natural connection between the inequality (3.1) and the functional (3.2).

Lemma 1. *Assume that $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type and u is of bounded variation on $[a,b]$. If $c\in(a,b)$, then*

$$\theta(f,u,x)(a,b)\geq\theta(f,u,x)(a,c)+\theta(f,u,x)(c,b) \tag{3.3}$$

for any $x\in[a,b]$, i.e., $\theta(f,u,x)$ is superadditive as a function of interval.

Proof. Observe that, for any $c\in(a,b)$, we have successively

$$\begin{aligned}\theta(f,u,x)(a,b)&=H\left(\max_{t\in[a,b]}|t-x|^{r}\right)\bigvee_{a}^{b}(u)-\left|\int_{a}^{b}[f(x)-f(t)]\,du(t)\right| \\ &=H\max\left\{\max_{t\in[a,c]}|t-x|^{r},\max_{t\in[c,b]}|t-x|^{r}\right\}\left(\bigvee_{a}^{c}(u)+\bigvee_{c}^{b}(u)\right) \\ &\qquad-\left|\int_{a}^{c}[f(x)-f(t)]\,du(t)+\int_{c}^{b}[f(x)-f(t)]\,du(t)\right|.\end{aligned} \tag{3.4}$$

Now, since

$$\begin{aligned}&H\max\left\{\max_{t\in[a,c]}|t-x|^{r},\max_{t\in[c,b]}|t-x|^{r}\right\}\left(\bigvee_{a}^{c}(u)+\bigvee_{c}^{b}(u)\right) \\ &\qquad\geq H\left[\max_{t\in[a,c]}|t-x|^{r}\bigvee_{a}^{c}(u)+\max_{t\in[c,b]}|t-x|^{r}\bigvee_{c}^{b}(u)\right]\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\left|\int_{a}^{c}[f(x)-f(t)]\,du(t)+\int_{c}^{b}[f(x)-f(t)]\,du(t)\right| \\ &\qquad\leq\left|\int_{a}^{c}[f(x)-f(t)]\,du(t)\right|+\left|\int_{c}^{b}[f(x)-f(t)]\,du(t)\right|\end{aligned} \tag{3.6}$$

then, by (3.4), we have

$$\begin{aligned}&\theta(f,u,x)(a,b) \\ &\geq H\max_{t\in[a,c]}|t-x|^{r}\bigvee_{a}^{c}(u)+H\max_{t\in[c,b]}|t-x|^{r}\bigvee_{c}^{b}(u) \\ &\quad-\left|\int_{a}^{c}[f(x)-f(t)]\,du(t)\right|-\left|\int_{c}^{b}[f(x)-f(t)]\,du(t)\right| \\ &=\theta(f,u,x)(a,c)+\theta(f,u,x)(c,b)\end{aligned}$$

and the statement is proved. □

Corollary 1. *Assume that* f *and* u *are as above. If* $x \in [c,d] \subset [a,b]$ *, then*

$$\theta(f,u,x)(a,b) \geq \theta(f,u,x)(c,d)$$

or, equivalently,

$$H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b (u) - \left|f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t)\right| \geq H\left[\frac{1}{2}(d-c)+\left|x-\frac{c+d}{2}\right|\right]^r \bigvee_c^d (u) - \left|f(x)[u(d)-u(c)] - \int_c^d f(t)\,du(t)\right|. \tag{3.7}$$

As in the general case presented above, we consider the following composite functionals that can be attached to $\theta(f,u,x)$

$$\Phi(f,u,x)(a,b) := \left[\frac{\theta(f,u,x)(a,b)}{b-a}\right]^{(b-a)},$$

$$\Upsilon(f,u,x)(a,b) := \frac{(b-a)^2}{\theta(f,u,x)(a,b)},$$

provided the denominator is not zero, and the families of functionals

$$\Omega_q(f,u,x)(a,b) := (b-a)^{1-q}[\theta(f,u,x)(a,b)]^q, q \in (0,1)$$

and

$$\Lambda_{p,q}(f,u,x)(a,b) := (b-a)^{\frac{p-q}{p}}[\theta(f,u,x)(a,b)]^q, p \geq q \geq 0, p \geq 1.$$

Proposition 1. *Assume that* $f:[a,b] \to \mathbb{R}$ *is of* $r-H-$*Hölder type and* u *is of bounded variation on* $[a,b]$ *. For* $x \in (a,b)$ *, the functional* $\Phi(f,u,x)$ *is supermultiplicative,* $\Upsilon(f,u,x)$ *is subadditive and* $\Omega_q(f,u,x)$ *and* $\Lambda_{p,q}(f,u,x)$ *are superadditive as functions of interval.*

The proof follows from Lemma 1 and the results from Introduction and Section 2. The details are omitted.

4. Applications for the Generalised Trapezoidal Formula

In [7], in order to approximate the Stieltjes integral $\int_a^b f(t)\,du(t)$ with the generalised trapezoidal rule $[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)$, where f is a function of bounded variation while u is continuous on $[a,b]$, the authors considered the *Generalised Trapezoidal error functional*

$$T(f,u;x,[a,b]) := [u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) - \int_a^b f(t)\,du(t)$$

and showed that

$$|T(f,u;x,[a,b])| \leq \max_{t\in[a,b]} |u(t)-u(x)| \bigvee_a^b (f). \tag{4.1}$$

Now, if $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type, where $r\in(0,1]$ and $H>0$ are given and u is of bounded variation on $[a,b]$, then by (4.1) we have the *generalised trapezoid inequality*

$$|T(f,u;x,[a,b])|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(u) \tag{4.2}$$

for any $x\in[a,b]$.

Now, we can define the following functional:

$$\eta(f,u,x)(a,b):=H\max_{t\in[a,b]}|t-x|^r\bigvee_a^b(u)-|T(f,u;x,[a,b])| \tag{4.3}$$

where f,u,x,a,b,r and H are as above.

Lemma 2. *Assume that $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type and u is of bounded variation on $[a,b]$. If $c\in(a,b)$, then*

$$\eta(f,u,x)(a,b)\ge\eta(f,u,x)(a,c)+\eta(f,u,x)(c,b) \tag{4.4}$$

for any $x\in[a,b]$, i.e. $\eta(f,u,x)$ is superadditive as a function of interval.

The proof is similar to the one from Lemma 1 by observing that

$$T(f,u;x,[a,b])=T(f,u;x,[a,c])+T(f,u;x,[c,b])$$

for $x,c\in[a,b]$ and we omit the details.

Corollary 2. *Assume that f and u are as above. If $x\in[c,d]\subset[a,b]$, then*

$$\eta(f,u,x)(a,b)\ge\eta(f,u,x)(c,d)$$

or, equivalently,

$$\begin{aligned} &H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(u) \\ &\quad-\left|[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a)-\int_a^b f(t)\,du(t)\right| \\ &\ge H\left[\frac{1}{2}(d-c)+\left|x-\frac{c+d}{2}\right|\right]^r\bigvee_c^d(u) \\ &\quad-\left|[u(d)-u(x)]f(d)+[u(x)-u(c)]f(c)-\int_c^d f(t)\,du(t)\right|. \end{aligned} \tag{4.5}$$

As in the general case presented above, we consider the following composite functionals that can be attached to $\eta(f,u,x)$

$$F(f,u,x)(a,b):=\left[\frac{\eta(f,u,x)(a,b)}{b-a}\right]^{(b-a)},$$

$$\Delta(f,u,x)(a,b):=\frac{(b-a)^2}{\eta(f,u,x)(a,b)},$$

provided the denumerator is not zero, and the families of functionals

$$\Xi_q(f,u,x)(a,b):=(b-a)^{1-q}[\eta(f,u,x)(a,b)]^q,\ q\in(0,1)$$

and

$$\Pi_{p,q}(f,u,x)(a,b) := (b-a)^{\frac{p-q}{p}} [\eta(f,u,x)(a,b)]^q, p \geq q \geq 0, p \geq 1.$$

Proposition 2. *Assume that $f:[a,b]\to\mathbb{R}$ is of $r-H-$Hölder type and u is of bounded variation on $[a,b]$. For $x\in(a,b)$, the functional $F(f,u,x)$ is supermultiplicative, $\Delta(f,u,x)$ is subadditive and $\Xi_q(f,u,x)$ and $\Pi_{p,q}(f,u,x)$ are superadditive as functions of interval.*

5. Applications for Means

If $g:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$ then $u(t)=\int_a^t g(s)\,ds$ is differentiable on (a,b) and the functionals Ψ, Υ, Ω_q and $\Lambda_{p,q}$ become

$$\tilde{\Psi}(f,g;[a,b]) := \max_{t\in[a,b]}|f(t)|\cdot\int_a^b|g(t)|\,dt - \left|\int_a^b f(t)\,g(t)\,dt\right|, \tag{5.1}$$

$$\tilde{\Upsilon}(f,g;[a,b]) := \frac{b-a}{\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\cdot\int_a^b|g(t)|\,dt - \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|} = \frac{(b-a)^2}{\tilde{\Psi}(f,g;[a,b])}. \tag{5.2}$$

$$\tilde{\Omega}_q(f,g;[a,b]) := (b-a)^{1-q}\left[\tilde{\Psi}(f,g;[a,b])\right]^q = (b-a)\left[\max_{t\in[a,b]}|f(t)|\frac{1}{b-a}\cdot\int_a^b|g(t)|\,dt - \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|\right]^q \tag{5.3}$$

and

$$\tilde{\Lambda}_{p,q}(f,g;[a,b]) := (b-a)^{\frac{p-q}{p}}\tilde{\Psi}^q(f,g;[a,b]) = (b-a)^{\frac{p-q+pq}{p}}\left[\max_{t\in[a,b]}|f(t)|\cdot\frac{1}{b-a}\int_a^b|g(t)|\,dt - \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\right|\right]^q. \tag{5.4}$$

Obviously $\tilde{\Psi}$ remains *superadditive and monotonic nondecreasing* as a function of an interval while $\tilde{\Upsilon}$ inherits the *subadditivity property* of Υ. Also, any member of the families of functionals $\tilde{\Omega}_q, q\in(0,1)$ and $\tilde{\Lambda}_{p,q}, p\geq q\geq 0, p\geq 1$ is *superadditive and monotonic nondecreasing as a function of interval.*

Let us recall the following means:

$$\begin{aligned}
\text{Arithmetic mean} &\quad : \quad A(a,b) = \frac{a+b}{2}, \\
\text{Geometric mean} &\quad : \quad G(a,b) = \sqrt{ab}, \\
\text{Harmonic mean} &\quad : \quad H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}, \\
\text{Logarithmic mean} &\quad : \quad L(a,b) = \frac{b-a}{\ln b - \ln a}, \ b \neq a; \\
\text{Identric mean:} &\quad\quad\ \ I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \ b \neq a; \\
p\text{-Logarithmic mean:} &\quad\quad\ \ L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \\
p &\in \mathbb{R}\backslash\{-1,0\}, b \neq a;
\end{aligned}$$

with $a,b>0$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities:

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b). \tag{5.5}$$

If we consider $u(t) = t^p$, $p \in \mathbb{R}\backslash\{-1,0\}$, $u : [a,b] \to \mathbb{R}$, $0<a<b$, then obviously

$$\frac{1}{b-a}\int_a^b u(t)\,dt = L_p^p(a,b).$$

If $u(t) = \frac{1}{t}$, $t \in [a,b]$, $0<a<b$, then

$$\frac{1}{b-a}\int_a^b u(t)\,dt = \frac{1}{L(a,b)},$$

while for $u(t) = \ln t$, $t \in [a,b]$, $0<a<b$, we have

$$\frac{1}{b-a}\int_a^b u(t)\,dt = \ln[I(a,b)].$$

If we choose above $g(t) = \frac{1}{t}$, $f(t) = t^p$ with $p \in \mathbb{R}\backslash\{0,1\}$ and observing that $0<a<b$, we have:

$$\max_{tse[a,b]}|f(t)| = \max\{a^p,b^p\}, \qquad \int_a^b g(t)\,dt = \frac{b-a}{L(a,b)}$$

and

$$\int_a^b f(t)\,g(t)\,dt = L_{p-1}^{p-1}(a,b)(b-a),$$

then we deduce that

$$v_p(a,b) := \left[\frac{\max\{a^p,b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)\right](b-a), \qquad p \in \mathbb{R}\backslash\{0,1\}. \tag{5.6}$$

is *superadditive and monotonic nondecreasing* as a function of interval while

$$z_p(a,b) := \frac{b-a}{\frac{\max\{a^p,b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)}, \qquad p \in \mathbb{R}\backslash\{0,1\} \tag{5.7}$$

is *subadditive* as a function of interval.

Finally, if we consider the families of functionals

$$y_{p,q}(a,b) := (b-a)\left[\frac{\max\{a^p,b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)\right]^q, \tag{5.8}$$

where $p \in \mathbb{R}\backslash\{0,1\}, q \in (0,1)$ and

$$u_{p,r,q}(a,b) = (b-a)^{\frac{r-q+rq}{r}}\left[\frac{\max\{a^p,b^p\}}{L(a,b)} - L_{p-1}^{p-1}(a,b)\right]^q, \tag{5.9}$$

where $p \in \mathbb{R}\backslash\{0,1\}, r \geq q \geq 0$ and $r \geq 1$, then we can conclude that each functional $y_{p,q}$ is *superadditive and monotonic nondecreasing as a function of interval* for any $p \in \mathbb{R}\backslash\{0,1\}, q \in (0,1)$. Also, each functional $u_{p,r,q}$ is *superadditive and monotonic nondecreasing as a function of interval* for any $p \in \mathbb{R}\backslash\{0,1\}$ and $r \geq q \geq 0$ and $r \geq 1$.

Similar results may be stated for other choices of f and g. However, the details are omitted.

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