Quasilinearity of Some Composite Functionals Associated to Schwarz's Inequality for Inner Products

S.S. Dragomir and Anca C. Goşa

Abstract. The quasilinearity of certain composite functionals associated to the celebrated Schwarz's inequality for inner products is investigated . Applications for operators in Hilbert spaces are given as well.

1. Introduction

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, nonnegative forms on X, i.e., the mapping $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions:

(i) $\langle x,x\rangle\geq 0$ for all $x\in X$;
(ii) $\langle \alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$ for all $x,y\in X$ and $\alpha,\beta\in\mathbb{K}$;
(iii) $\langle y,x\rangle=\overline{\langle x,y\rangle}$ for all $x,y\in X$.

If $\langle\cdot,\cdot\rangle\in\mathcal{H}(X)$, then the functional $\|\cdot\|=\langle\cdot,\cdot\rangle^{1/2}$ is a semi-norm on X and the following version of Schwarz's inequality holds:

$$\|x\|^2\|y\|^2\geq|\langle x,y\rangle|^2 \tag{1.1}$$

for each $x,y\in H$.

In addition, if $\langle\cdot,\cdot\rangle$ is an inner product on X, i.e., satisfies the condition

(iv) $\langle x,x\rangle=0$ only if $x=0$;

then the equality case holds in (1.1) if and only if the vectors x and y are linearly dependent.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. Also, we can introduce on $\mathcal{H}(X)$ the following *binary relation* [**5**]

$$\langle\cdot,\cdot\rangle_2\geq\langle\cdot,\cdot\rangle_1 \quad\text{if and only if}\quad \|x\|_2\geq\|x\|_1 \quad\text{for any}\quad x\in H. \tag{1.2}$$

This is an *order relation* on $\mathcal{H}(X)$, see [**5**].

For some classical results related to the celebrated Schwarz's inequality, see [**7**], [**9**], [**11**], [**12**] and [**13**].

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For recent results, see [**1**], [**2**], [**3**], [**4**], [**8**], [**10**] and [**14**] and the references therein.

2. Some Functionals Related to Schwarz's Inequality

Consider the following functional [**5**]:

$$\delta : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \qquad \delta(\langle \cdot,\cdot \rangle ; x, y) := \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2, \tag{2.1}$$

which is closely related to the Schwarz inequality in (1.1).

Theorem 1 ([**5**]). *The functional $\delta(\cdot; x, y)$ is nonnegative, superadditive, monotonic nondecreasing and quadratic positive homogeneous on $\mathcal{H}(X)$.*

The nonnegativity of $\delta(\cdot; x, y)$ is in fact the Schwarz inequality (1.1). The superadditivity property is translated in the fact that

$$\begin{aligned} \delta(\langle \cdot,\cdot \rangle_1 + \langle \cdot,\cdot \rangle_2 ; x, y) &:= \left(\|x\|_1^2 + \|x\|_2^2\right)\left(\|y\|_1^2 + \|y\|_2^2\right) - |\langle x, y\rangle_1 + \langle x, y\rangle_2|^2 \\ &\geq \|x\|_1^2 \|y\|_1^2 + \|x\|_2^2 \|y\|_2^2 - |\langle x, y\rangle_1|^2 - |\langle x, y\rangle_2|^2 \\ &= \delta(\langle \cdot,\cdot \rangle_1 ; x, y) + \delta(\langle \cdot,\cdot \rangle_2 ; x, y) \end{aligned} \tag{2.2}$$

for any $\langle \cdot,\cdot \rangle_1, \langle \cdot,\cdot \rangle_2 \in \mathcal{H}(X)$ and $x, y \in X$.

If $\langle \cdot,\cdot \rangle_2 \geq \langle \cdot,\cdot \rangle_1$ in the sense specified in (1.2), then the monotonicity property mentioned in Theorem 1 becomes the inequality

$$\begin{aligned} \delta(\langle \cdot,\cdot \rangle_2 ; x, y) &= \|x\|_2^2 \|y\|_2^2 - |\langle x, y\rangle_2|^2 \\ &\geq \|x\|_1^2 \|y\|_1^2 - |\langle x, y\rangle_1|^2 = \delta(\langle \cdot,\cdot \rangle_1 ; x, y) \end{aligned} \tag{2.3}$$

for any $x, y \in X$. This inequality is of interest due to the fact that it creates the possibility to provide various refinements for the Schwarz inequality for inner products as pointed out below.

The quadratic positive homogeneous property means that $\delta(\alpha \langle \cdot,\cdot \rangle ; x, y) = \alpha^2 \delta(\langle \cdot,\cdot \rangle ; x, y)$ for any $\alpha \in \mathbb{R}_+$.

As a natural corollary of the above we have the result:

Corollary 1. *Let $\langle \cdot,\cdot \rangle_1, \langle \cdot,\cdot \rangle_2 \in \mathcal{H}(X)$ be such that there exists the constants $M > m > 0$ with the property that*

$$M\|x\|_1 \geq \|x\|_2 \geq m\|x\|_1 \tag{2.4}$$

for any $x \in X$, meaning that the seminorms $\|\cdot\|_2$ and $\|\cdot\|_1$ are equivalent. Then we have the inequalities

$$\begin{aligned} M^4\left[\|x\|_1^2 \|y\|_1^2 - |\langle x, y\rangle_1|^2\right] &\geq \|x\|_2^2 \|y\|_2^2 - |\langle x, y\rangle_2|^2 \\ &\geq m^4\left[\|x\|_1^2 \|y\|_1^2 - |\langle x, y\rangle_1|^2\right] \end{aligned} \tag{2.5}$$

for any $x, y \in X$.

Another functional that can be associated with Schwarz's inequality is the following one

$$\beta : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \qquad \beta(\langle \cdot,\cdot \rangle ; x, y) := \left(\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2\right)^{1/2}.$$

The properties of this functional have been established in 1994 by B. Mond and the author:

THEOREM 2 ([**6**]). *The functional $\beta(\cdot;x,y)$ is nonnegative, superadditive, monotonic nondecreasing and positive homogeneous on $\mathcal{H}(X)$.*

One can realize that the superadditivity property of $\beta(\cdot;x,y)$ implies the same property for $\delta(\cdot;x,y)$ and therefore provides an alternative proof for Theorem 1.

A different functional associated with the order one version of Schwarz's inequality, namely

$$\|x\|\,\|y\| \geq |\langle x,y\rangle|,$$

has been also considered in [**5**]. The definition of this functional is

$$\sigma : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \qquad \sigma(\langle\cdot,\cdot\rangle;x,y) := \|x\|\,\|y\| - |\langle x,y\rangle|$$

and its properties are incorporated in

THEOREM 3 ([**5**]). *The functional $\sigma(\cdot;x,y)$ is nonnegative, superadditive, monotonic nondecreasing and positive homogeneous on $\mathcal{H}(X)$.*

As a consequence of this result that may be useful for applications we have:

COROLLARY 2. *Let $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2 \in \mathcal{H}(X)$ be such that there exists the constants $M > m > 0$ with the property (2.4). Then we have the inequalities*

$$M^2\left[\|x\|_1\|y\|_1 - |\langle x,y\rangle_1|\right] \geq \|x\|_2\|y\|_2 - |\langle x,y\rangle_2| \geq m^2\left[\|x\|_1\|y\|_1 - |\langle x,y\rangle_1|\right] \tag{2.6}$$

for any $x,y \in X$.

Motivated by the above results, we investigate in the present paper some composite functionals that are related to the above ones, establish their superadditivity and monotonicity properties and apply them for bounded linear operators in Hilbert spaces.

3. Some Composite Functionals and Their Properties

Now, assume that $\psi : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+$ is a nonnegative, superadditive and r-positive homogeneous on $\mathcal{H}(X)$, meaning that

$$\psi(\alpha\langle\cdot,\cdot\rangle;x,y) = \alpha^r\psi(\langle\cdot,\cdot\rangle;x,y)$$

for any $\alpha \geq 0$.

For $e,x,y \in X$ with $e \neq 0$ and $p,q \geq 1$ we consider the composite functional $\xi_{e,p,q}(\cdot;x,y) : \mathcal{H}(X) \to [0,\infty)$ given by

$$\xi_{e,p,q}(\langle\cdot,\cdot\rangle;x,y) := \|e\|^{2\left(1-\frac{1}{p}\right)q}\,\psi^q(\langle\cdot,\cdot\rangle;x,y), \tag{3.1}$$

where ψ is as above.

The following result holds:

THEOREM 4. *Assume that $\psi : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+$ is a nonnegative, superadditive and r-positive homogeneous on $\mathcal{H}(X)$, then the functional $\xi_{e,p,q}(\cdot;x,y)$ defined by (3.1) is superadditive, monotonic nondecreasing and $q\left(r+1-\frac{1}{p}\right)$–positive homogeneous on $\mathcal{H}(X)$.*

PROOF. First of all we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p \geq (\leq)\, \alpha^p + \beta^p \tag{3.2}$$

for any $\alpha,\beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0,\infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (3.2).

For $p \in (0,1)$ we have that f_p is strictly decreasing on $[0,\infty)$ which proves the second case in (*3.2*).

We will prove first the case $q = 1$.

Let $x, y \in X$. Since $\psi(\cdot; x, y)$ is superadditive and $p \geq 1$, then we have by (3.2) that

$$\begin{aligned} \psi^p\left(\langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2 ; x, y\right) &\geq \left[\psi\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \psi\left(\langle\cdot,\cdot\rangle_2 ; x, y\right)\right]^p \\ &\geq \psi^p\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \psi^p\left(\langle\cdot,\cdot\rangle_2 ; x, y\right) \end{aligned} \tag{3.3}$$

for any $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2 \in \mathcal{H}(X)$.

Let $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2 \in \mathcal{H}(X)$. If $e \in X, e \neq 0$ is such that either $\langle e,e\rangle_1 = 0$ or $\langle e,e\rangle_2 = 0$, then the superadditivity property is trivially satisfied, so we can assume further that $\langle e,e\rangle_1 \neq 0$ and $\langle e,e\rangle_2 \neq 0$. Therefore, by (3.3) we have that

$$\begin{aligned} \frac{\psi^p\left(\langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2} &\geq \frac{\psi^p\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \psi^p\left(\langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2} \\ &= \frac{\langle e,e\rangle_1 \cdot \frac{\psi^p\left(\langle\cdot,\cdot\rangle_1 ; x,y\right)}{\langle e,e\rangle_1} + \langle e,e\rangle_2 \cdot \frac{\psi^p\left(\langle\cdot,\cdot\rangle_2 ; x,y\right)}{\langle e,e\rangle_2}}{\langle e,e\rangle_1 + \langle e,e\rangle_2} \\ &= \frac{\langle e,e\rangle_1 \cdot \left[\frac{\psi\left(\langle\cdot,\cdot\rangle_1 ; x,y\right)}{\langle e,e\rangle_1^{1/p}}\right]^p + \langle e,e\rangle_2 \cdot \left[\frac{\psi\left(\langle\cdot,\cdot\rangle_2 ; x,y\right)}{\langle e,e\rangle_2^{1/p}}\right]^p}{\langle e,e\rangle_1 + \langle e,e\rangle_2}. \end{aligned} \tag{3.4}$$

Since for $p \geq 1$ the power function is convex, then

$$\begin{aligned} &\frac{\langle e,e\rangle_1 \cdot \left[\frac{\psi\left(\langle\cdot,\cdot\rangle_1 ; x,y\right)}{\langle e,e\rangle_1^{1/p}}\right]^p + \langle e,e\rangle_2 \cdot \left[\frac{\psi\left(\langle\cdot,\cdot\rangle_2 ; x,y\right)}{\langle e,e\rangle_2^{1/p}}\right]^p}{\langle e,e\rangle_1 + \langle e,e\rangle_2} \\ &\geq \left[\frac{\langle e,e\rangle_1 \cdot \frac{\psi\left(\langle\cdot,\cdot\rangle_1 ; x,y\right)}{\langle e,e\rangle_1^{1/p}} + \langle e,e\rangle_2 \cdot \frac{\psi\left(\langle\cdot,\cdot\rangle_2 ; x,y\right)}{\langle e,e\rangle_2^{1/p}}}{\langle e,e\rangle_1 + \langle e,e\rangle_2}\right]^p \\ &= \left[\frac{\langle e,e\rangle_1^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \langle e,e\rangle_2^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2}\right]^p. \end{aligned} \tag{3.5}$$

By combining (3.4) with (3.5) we get

$$\frac{\psi^p\left(\langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2} \geq \left[\frac{\langle e,e\rangle_1^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \langle e,e\rangle_2^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2}\right]^p,$$

which, by taking the power $1/p$ is equivalent with

$$\frac{\psi\left(\langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2 ; x, y\right)}{\left[\langle e,e\rangle_1 + \langle e,e\rangle_2\right]^{1/p}} \geq \frac{\langle e,e\rangle_1^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_1 ; x, y\right) + \langle e,e\rangle_2^{1-\frac{1}{p}} \psi\left(\langle\cdot,\cdot\rangle_2 ; x, y\right)}{\langle e,e\rangle_1 + \langle e,e\rangle_2}$$

showing that

$$\delta_{e,p,1}\left(\langle\cdot,\cdot\rangle_1+\langle\cdot,\cdot\rangle_2;x,y\right)\geq\delta_{e,p,1}\left(\langle\cdot,\cdot\rangle_1;x,y\right)+\delta_{e,p,2}\left(\langle\cdot,\cdot\rangle_2;x,y\right)$$

for any $\langle\cdot,\cdot\rangle_1,\langle\cdot,\cdot\rangle_2\in\mathcal{H}(X)$.

Now, observe that

$$\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle;x,y\right)=\left[\xi_{e,p,1}\left(\langle\cdot,\cdot\rangle;x,y\right)\right]^q$$

for any $q>1$ and by the elementary inequality (3.2) we have that

$$\begin{aligned}\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_1+\langle\cdot,\cdot\rangle_2;x,y\right)&=\left[\xi_{e,p,1}\left(\langle\cdot,\cdot\rangle_1+\langle\cdot,\cdot\rangle_2;x,y\right)\right]^q\\&\geq\left[\xi_{e,p,1}\left(\langle\cdot,\cdot\rangle_1;x,y\right)+\xi_{e,p,2}\left(\langle\cdot,\cdot\rangle_2;x,y\right)\right]^q\\&\geq\xi^q_{e,p,1}\left(\langle\cdot,\cdot\rangle_1;x,y\right)+\xi^q_{e,p,2}\left(\langle\cdot,\cdot\rangle_2;x,y\right)\\&=\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_1;x,y\right)+\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_2;x,y\right)\end{aligned}\tag{3.6}$$

for any $\langle\cdot,\cdot\rangle_1,\langle\cdot,\cdot\rangle_2\in\mathcal{H}(X)$ and the superadditivity of the functional $\xi_{e,p,q}(\cdot;x,y)$ is proven.

Further, assume that $\langle\cdot,\cdot\rangle_2\geq\langle\cdot,\cdot\rangle_1$. Since $\langle\cdot,\cdot\rangle_{2,1}:=\langle\cdot,\cdot\rangle_2-\langle\cdot,\cdot\rangle_1$ is on its turn an Hermitian functional, then by superadditivity property of $\xi_{e,p,q}(\cdot;x,y)$ we have

$$\begin{aligned}\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_2;x,y\right)&=\xi_{e,p,q}\Big(\langle\cdot,\cdot\rangle_1+\langle\cdot,\cdot\rangle_{2,1};x,y\Big)\\&\geq\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_1;x,y\right)+\xi_{e,p,q}\Big(\langle\cdot,\cdot\rangle_{2,1};x,y\Big)\end{aligned}$$

which, obviously, implies that

$$\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_2;x,y\right)-\xi_{e,p,q}\left(\langle\cdot,\cdot\rangle_1;x,y\right)\geq\xi_{e,p,q}\Big(\langle\cdot,\cdot\rangle_{2,1};x,y\Big)\geq0$$

giving the desired monotonicity result.

The $q\left(r+1-\frac{1}{p}\right)-$positive homogeneity of the functional is obvious by (3.1) and the proof is complete. □

Remark 1. *If for $q=p\geq1$ we consider the functional*

$$\xi_{e,p}\left(\langle\cdot,\cdot\rangle;x,y\right):=\langle e,e\rangle^{p-1}\psi^p\left(\langle\cdot,\cdot\rangle;x,y\right),\tag{3.7}$$

then, by Theorem 4, we have that $\xi_{e,p}(\cdot;x,y)$ is superadditive, monotonic nondecreasing and $(p(r+1)-1)-$positive homogeneous on $\mathcal{H}(X)$.

The following result for equivalent seminorms is of interest for applications:

Corollary 3. *Assume that $\psi:\mathcal{H}(X)\times X^2\to\mathbb{R}_+$ is a nonnegative, superadditive and r-positive homogeneous on $\mathcal{H}(X)$ and $\langle\cdot,\cdot\rangle_1,\langle\cdot,\cdot\rangle_2\in\mathcal{H}(X)$ are such that there exists the constants $M>m>0$ with the property that*

$$M\|x\|_1\geq\|x\|_2\geq m\|x\|_1$$

for any $x\in X$. Then for $p\geq1$ and $e\in X$ with $\|e\|_1>0$ we have the inequalities

$$\begin{aligned}M^{2q\left(r+1-\frac{1}{p}\right)}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{2\left(1-\frac{1}{p}\right)}&\psi\left(\langle\cdot,\cdot\rangle_1;x,y\right)\\&\geq\psi\left(\langle\cdot,\cdot\rangle_2;x,y\right)\\&\geq m^{2q\left(r+1-\frac{1}{p}\right)}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{2\left(1-\frac{1}{p}\right)}\psi\left(\langle\cdot,\cdot\rangle_1;x,y\right).\end{aligned}\tag{3.8}$$

Proof. Utilising the monotonicity and $q\left(r+1-\frac{1}{p}\right)-$positive homogeneity of the functional $\xi_{e,p,q}\left(\cdot;x,y\right)$ we have

$$\begin{aligned} M^{2q\left(r+1-\frac{1}{p}\right)}\left\|e\right\|_{1}^{2q\left(1-\frac{1}{p}\right)}\psi^{q}\left(\left\langle\cdot,\cdot\right\rangle_{1};x,y\right) &\geq \left\|e\right\|_{2}^{2\left(1-\frac{1}{p}\right)q}\psi^{q}\left(\left\langle\cdot,\cdot\right\rangle_{2};x,y\right) \\ &\geq m^{2q\left(r+1-\frac{1}{p}\right)}\left\|e\right\|_{1}^{2q\left(1-\frac{1}{p}\right)}\psi^{q}\left(\left\langle\cdot,\cdot\right\rangle_{1};x,y\right) \end{aligned}$$

which, by taking the power $1/q$ we deduce the desired result (3.1). □

Now, returning back to our functionals associated to Schwarz's inequality, we can state the following result:

Theorem 5. *For $e,x,y\in X$ with $e\neq 0$ and $p,q\geq 1$ we consider the composite functionals*

$$\delta_{e,p,q}\left(\left\langle\cdot,\cdot\right\rangle;x,y\right) := \left\|e\right\|^{2\left(1-\frac{1}{p}\right)q}\left[\left\|x\right\|^{2}\left\|y\right\|^{2}-\left|\left\langle x,y\right\rangle\right|^{2}\right]^{q},$$

$$\beta_{e,p,q}\left(\left\langle\cdot,\cdot\right\rangle;x,y\right) := \left\|e\right\|^{2\left(1-\frac{1}{p}\right)q}\left[\left\|x\right\|^{2}\left\|y\right\|^{2}-\left|\left\langle x,y\right\rangle\right|^{2}\right]^{q/2}$$

and

$$\delta_{e,p,q}\left(\left\langle\cdot,\cdot\right\rangle;x,y\right) := \left\|e\right\|^{2\left(1-\frac{1}{p}\right)q}\left[\left\|x\right\|\left\|y\right\|-\left|\left\langle x,y\right\rangle\right|\right]^{q}.$$

Then these functionals are superadditive and monotonic nondecreasing. The first functional is $q\left(3-\frac{1}{p}\right)-$positive homogeneous while the second and the third are $q\left(2-\frac{1}{p}\right)-$positive homogeneous on $\mathcal{H}\left(X\right)$.

The proof follows from Theorem 4.

As applications in providing some refinements for the celebrated Schwarz's inequality we can state the following inequalities:

Corollary 4. *Let $\left\langle\cdot,\cdot\right\rangle_{1},\left\langle\cdot,\cdot\right\rangle_{2}\in\mathcal{H}\left(X\right)$ be such that there exists the constants $M>m>0$ with the property (2.4). If $e\in X$ with $\left\|e\right\|_{1}>0$, then for $p\geq 1$ we have the inequalities*

$$\begin{aligned} M^{2q\left(3-\frac{1}{p}\right)}\left(\frac{\left\|e\right\|_{1}}{\left\|e\right\|_{2}}\right)^{2\left(1-\frac{1}{p}\right)}\left[\left\|x\right\|_{1}^{2}\left\|y\right\|_{1}^{2}-\left|\left\langle x,y\right\rangle_{1}\right|^{2}\right] &\geq \left\|x\right\|_{2}^{2}\left\|y\right\|_{2}^{2}-\left|\left\langle x,y\right\rangle_{2}\right|^{2} \\ &\geq m^{2q\left(3-\frac{1}{p}\right)}\left(\frac{\left\|e\right\|_{1}}{\left\|e\right\|_{2}}\right)^{2\left(1-\frac{1}{p}\right)}\left[\left\|x\right\|_{1}^{2}\left\|y\right\|_{1}^{2}-\left|\left\langle x,y\right\rangle_{1}\right|^{2}\right] \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} M^{2q\left(2-\frac{1}{p}\right)}\left(\frac{\left\|e\right\|_{1}}{\left\|e\right\|_{2}}\right)^{2\left(1-\frac{1}{p}\right)}\left[\left\|x\right\|_{1}^{2}\left\|y\right\|_{1}^{2}-\left|\left\langle x,y\right\rangle_{1}\right|^{2}\right]^{1/2} &\geq \left[\left\|x\right\|_{2}^{2}\left\|y\right\|_{2}^{2}-\left|\left\langle x,y\right\rangle_{2}\right|^{2}\right]^{1/2} \\ &\geq m^{2q\left(2-\frac{1}{p}\right)}\left(\frac{\left\|e\right\|_{1}}{\left\|e\right\|_{2}}\right)^{2\left(1-\frac{1}{p}\right)}\left[\left\|x\right\|_{1}^{2}\left\|y\right\|_{1}^{2}-\left|\left\langle x,y\right\rangle_{1}\right|^{2}\right]^{1/2} \end{aligned} \tag{3.10}$$

and

$$M^{2q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|_1\|y\|_1-|\langle x,y\rangle_1|\right] \geq \|x\|_2\|y\|_2-|\langle x,y\rangle_2| \geq m^{2q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|_1}{\|e\|_2}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|_1\|y\|_1-|\langle x,y\rangle_1|\right] \tag{3.11}$$

respectively, where $x,y\in X$.

4. Applications for Operators

Denote by $\mathcal{B}(H)$ the Banach algebra of bounded linear operator acting on the Hilbert space H. We recall that a selfadjoint operator $P\in\mathcal{B}(H)$ is *nonnegative* if $\langle Px,x\rangle\geq 0$ for any $x\in H$. P is called *positive* if it is nonnegative and $\langle Px,x\rangle=0$ implies that $x=0$ and *positive definite* with the constant $\gamma>0$ if $\langle Px,x\rangle\geq\gamma\|x\|^2$ for any $x\in H$. We denote by $\mathcal{P}(H)$ the convex cone of all selfadjoint nonnegative operators defined on H. If A,B are two selfadjoint operators on H we say that $A\geq B$ in the operator order of $\mathcal{B}(H)$ if $A-B\in\mathcal{P}(H)$.

Now, if $P\in\mathcal{P}(H)$ then the functional $\langle\cdot,\cdot\rangle_P:H\times H\to\mathbb{C}$, $\langle x,y\rangle_P:=\langle Px,y\rangle$ is an Hermitian form on H. If $A\geq B\geq 0$, then the corresponding Hermitian forms satisfy the property that $\langle\cdot,\cdot\rangle_A\geq\langle\cdot,\cdot\rangle_B$ in the sense of the definition from the introduction.

We can consider the following functionals

$$\Delta_{e,p,q}(\cdot;x,y),\Theta_{e,p,q}(\cdot;x,y),\Sigma_{e,p,q}(\cdot;x,y):\mathcal{P}(H)\to\mathbb{R}_+$$

given by

$$\Delta_{e,p,q}(P;x,y):=\langle Pe,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Px,x\rangle\langle Py,y\rangle-|\langle Px,y\rangle|^2\right]^q, \tag{4.1}$$

$$\Theta_{e,p,q}(P;x,y):=\langle Pe,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Px,x\rangle\langle Py,y\rangle-|\langle Px,y\rangle|^2\right]^{q/2}$$

and

$$\Sigma_{e,p,q}(P;x,y):=\langle Pe,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}-|\langle Px,y\rangle|\right]^q \tag{4.2}$$

respectively, where $e,x,y\in H, e\neq 0$ and $p,q\geq 1$.

The following result holds:

Proposition 1. *The functionals* $\Delta_{e,p,q}(\cdot;x,y),\Theta_{e,p,q}(\cdot;x,y),\Sigma_{e,p,q}(\cdot;x,y)$ *are superadditive and monotonic nondecreasing in the operator order of* $\mathcal{P}(H)$, *meaning, for instance that, if* $A\geq B\geq 0$, *then*

$$\Delta_{e,p,q}(A;x,y)\geq\Delta_{e,p,q}(B;x,y) \tag{4.3}$$

where $e,x,y\in H, e\neq 0$ *and* $p,q\geq 1$.

There are some particular inequalities of interest that can be stated by the use of this proposition:

Corollary 5. *If U is a selfadjoint operator with the property that $0 \le U \le I$ and if we denote by $\bar{U} := I - U$, then we have the inequalities*

$$\begin{aligned} \|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right]^q &\geq \langle Ue,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2\right]^q \\ &\quad + \left\langle \bar{U}e,e\right\rangle^{\left(1-\frac{1}{p}\right)q}\left[\left\langle \bar{U}x,x\right\rangle\left\langle \bar{U}y,y\right\rangle - \left|\left\langle \bar{U}x,y\right\rangle\right|^2\right]^q \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right]^{q/2} &\geq \langle Ue,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2\right]^{q/2} \\ &\quad + \left\langle \bar{U}e,e\right\rangle^{\left(1-\frac{1}{p}\right)q}\left[\left\langle \bar{U}x,x\right\rangle\left\langle \bar{U}y,y\right\rangle - \left|\left\langle \bar{U}x,y\right\rangle\right|^2\right]^{q/2} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|\|y\| - |\langle x,y\rangle|\right]^q &\geq \langle Ue,e\rangle^{\left(1-\frac{1}{p}\right)q}\left[\langle Ux,x\rangle^{1/2}\langle Uy,y\rangle^{1/2} - |\langle Ux,y\rangle|\right]^q \\ &\quad + \left\langle \bar{U}e,e\right\rangle^{\left(1-\frac{1}{p}\right)q}\left[\left\langle \bar{U}x,x\right\rangle^{1/2}\left\langle \bar{U}y,y\right\rangle^{1/2} - \left|\left\langle \bar{U}x,y\right\rangle\right|\right]^q \end{aligned} \tag{4.6}$$

respectively, where $e,x,y \in H, e \neq 0$ and $p,q \geq 1$.

Corollary 6. *Let V be a selfadjoint operator such that there exists the positive constants Φ and φ with the property that $\Phi I \geq V \geq \varphi I$ in the operator order, then*

$$\begin{aligned} \Phi^{q\left(3-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right] &\geq \langle Vx,x\rangle\langle Vy,y\rangle - |\langle Vx,y\rangle|^2 \\ &\geq \varphi^{q\left(3-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \Phi^{q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right]^{1/2} &\geq \left[\langle Vx,x\rangle\langle Vy,y\rangle - |\langle Vx,y\rangle|^2\right]^{1/2} \\ &\geq \varphi^{q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right]^{1/2} \end{aligned} \tag{4.8}$$

and

$$\Phi^{q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|\,\|y\|-|\langle x,y\rangle|\right] \tag{4.9}$$
$$\geq \langle Vx,x\rangle^{1/2}\langle Vy,y\rangle^{1/2}-|\langle Vx,y\rangle|$$
$$\geq \varphi^{q\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|^2}{\langle Ve,e\rangle}\right)^{\left(1-\frac{1}{p}\right)}\left[\|x\|\,\|y\|-|\langle x,y\rangle|\right]$$

respectively, where $x,y,e\in H$ *and* $e\neq 0$.

For two bounded linear operators A and B, we write that $A\tilde{\leq}B$ if $\|Ax\|\leq\|Bx\|$ for any $x\in H$. This is obviously equivalent with $A^*A\leq B^*B$ in the operator order of $\mathcal{B}(H)$. For any bounded linear operator A on H we can consider the Hermitian form

$$(x,y)_A:=\langle Ax,Ay\rangle=\langle A^*Ax,y\rangle=\langle x,y\rangle_{A^*A}.$$

We observe that $(\cdot,\cdot)_A\leq(\cdot,\cdot)_B$ in the sense of the definition from Introduction if and only if $A\tilde{\leq}B$.

We denote by $\mathcal{HB}(H)$ the convex cone of all Hermitian forms generated by the operators from $\mathcal{B}(H)$ as above.

Consider the following functionals

$$\tilde{\Delta}_{e,p,q}(\cdot;x,y),\tilde{\Theta}_{e,p,q}(\cdot;x,y),\tilde{\Sigma}_{e,p,q}(\cdot;x,y):\mathcal{HB}(H)\to\mathbb{R}_+$$

given by

$$\tilde{\Delta}_{e,p,q}\left((\cdot,\cdot)_T;x,y\right):=(e,e)_T^{\left(1-\frac{1}{p}\right)q}\left[(x,x)_T(y,y)_T-|(x,y)_T|^2\right]^q \tag{4.10}$$
$$=\|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^q,$$

$$\tilde{\Theta}_{e,p,q}\left((\cdot,\cdot)_T;x,y\right)=(e,e)_T^{\left(1-\frac{1}{p}\right)q}\left[(x,x)_T(y,y)_T-|(x,y)_T|^2\right]^{q/2} \tag{4.11}$$
$$=\|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^{q/2}$$

and

$$\tilde{\Sigma}_{e,p,q}\left((\cdot,\cdot)_T;x,y\right):=(e,e)_T^{\left(1-\frac{1}{p}\right)q}\left[(x,x)_T^{1/2}(y,y)_T^{1/2}-|(x,y)_T|\right]^q \tag{4.12}$$
$$=\|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|\,\|Ty\|-|\langle Tx,Ty\rangle|\right]^q$$

respectively, where $e,x,y\in H, e\neq 0$ and $p,q\geq 1$.

Utilising Theorem 4 we can state the following result concerning the properties of the functionals introduced at (4.10)-(4.12):

Proposition 2. *The functionals* $\tilde{\Delta}_{e,p,q}(\cdot;x,y),\tilde{\Theta}_{e,p,q}(\cdot;x,y)$ *and* $\tilde{\Sigma}_{e,p,q}(\cdot;x,y)$ *are superadditive and monotonic nondecreasing on* $\mathcal{HB}(H)$.

In terms of operators, the superadditivity property for the functional $\tilde{\Delta}_{e,p,q}(\cdot;x,y)$ can be translated as

$$\begin{aligned}(4.13)\quad &\left(\|Te\|^2+\|Ue\|^2\right)^{\left(1-\frac{1}{p}\right)q}\\ &\times\left[\left(\|Tx\|^2+\|Ux\|^2\right)\left(\|Ty\|^2+\|Uy\|^2\right)-|\langle Tx,Ty\rangle+\langle Ux,Uy\rangle|^2\right]^q\\ &\geq \|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^q\\ &\quad+\|Ue\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Ux\|^2\|Uy\|^2-|\langle Ux,Uy\rangle|^2\right]^q\end{aligned}$$

for any $T,U\in\mathcal{B}(H)$ and $x,y,e\in H$.

The monotonicity is as follows: if $T,U\in\mathcal{B}(H)$ are so that $\|Tx\|\geq\|Ux\|$ for any $x\in H$, then we have the inequality

$$\begin{aligned}&\|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^q\\ &\geq\|Ue\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Ux\|^2\|Uy\|^2-|\langle Ux,Uy\rangle|^2\right]^q\end{aligned}$$

for any $x,y,e\in H$, which, by taking the power $1/q$ is equivalent with

$$\begin{aligned}(4.14)\quad &\|Te\|^{2\left(1-\frac{1}{p}\right)}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]\\ &\geq\|Ue\|^{2\left(1-\frac{1}{p}\right)}\left[\|Ux\|^2\|Uy\|^2-|\langle Ux,Uy\rangle|^2\right]\end{aligned}$$

for any $x,y,e\in H$.

The following result that utilizes the superadditivity of the functionals above holds:

Proposition 3. *Let U,V be two bounded linear operators with the property that $U^*U+V^*V=I$. Then we have*

$$\begin{aligned}(4.15)\quad &\|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^q\\ &\geq\|Ue\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Ux\|^2\|Uy\|^2-|\langle Ux,Uy\rangle|^2\right]^q\\ &\quad+\|Ve\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Vx\|^2\|Vy\|^2-|\langle Vx,Vy\rangle|^2\right]^q\end{aligned}$$

and

$$\begin{aligned}(4.16)\quad &\|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^{q/2}\\ &\geq\|Ue\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Ux\|^2\|Uy\|^2-|\langle Ux,Uy\rangle|^2\right]^{q/2}\\ &\quad+\|Ve\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Vx\|^2\|Vy\|^2-|\langle Vx,Vy\rangle|^2\right]^{q/2}\end{aligned}$$

and

$$\begin{aligned}(4.17)\quad &\|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|\|y\|-|\langle x,y\rangle|\right]^q\\ &\geq\|Ue\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Ux\|\|Uy\|-|\langle Ux,Uy\rangle|\right]^q\\ &\quad+\|Ve\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Vx\|\|Vy\|-|\langle Vx,Vy\rangle|\right]^q\end{aligned}$$

respectively, for any $e, x, y \in H$.

Finally, the following result that holds for invertible bounded linear operators is of interest as well:

PROPOSITION 4. *Let* T *be an invertible bounded linear operator on* H. *Then we have*

$$\|T\|^{2\left(3-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right] \geq \|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2 \geq \left\|T^{-1}\right\|^{-2\left(3-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right] \tag{4.18}$$

and

$$\|T\|^{2\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^{1/2} \geq \left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^{1/2} \geq \left\|T^{-1}\right\|^{-2\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^{1/2} \tag{4.19}$$

and

$$\|T\|^{2\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|\|y\|-|\langle x,y\rangle|\right] \geq \|Tx\|\|Ty\|-|\langle Tx,Ty\rangle| \geq \left\|T^{-1}\right\|^{-2\left(2-\frac{1}{p}\right)}\left(\frac{\|e\|}{\|Te\|}\right)^{2\left(1-\frac{1}{p}\right)}\left[\|x\|\|y\|-|\langle x,y\rangle|\right] \tag{4.20}$$

respectively, for any $x, y, e \in H$ *and* $e \neq 0$.

PROOF. Since T is invertible, then we have

$$\|T\|\|x\| \geq \|Tx\| \geq \left\|T^{-1}\right\|^{-1}\|x\|$$

for any $x \in H$.

Utilising the monotonicity property of the functional $\tilde{\Delta}_{e,p,q}(\cdot; x, y)$, we have

$$\|T\|^{2\left(3-\frac{1}{p}\right)q}\|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^q \geq \|Te\|^{2\left(1-\frac{1}{p}\right)q}\left[\|Tx\|^2\|Ty\|^2-|\langle Tx,Ty\rangle|^2\right]^q \geq \left\|T^{-1}\right\|^{-2\left(3-\frac{1}{p}\right)q}\|e\|^{2\left(1-\frac{1}{p}\right)q}\left[\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\right]^q$$

and by taking the power $1/q$ we deduce the desired inequality (4.18).

A similar approach for $\tilde{\Theta}_{e,p,q}(\cdot; x, y)$ and $\tilde{\Sigma}_{e,p,q}(\cdot; x, y)$ will produce the other two inequalities. □

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Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/dragomir/

College 12, Reşiţa, Jud, Caraş-Severin, Romania

E-mail address: ancagosa@yahoo.com