THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS

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Abstract. Some Hermite-Hadamard's type inequalities for operator convex functions of selfadjoint operators in Hilbert spaces are given. Applications for particular cases of interest are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x) dx < (b-a) \frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [12]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [17].

E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D.S. Mitrinović found Hermite's note in *Mathesis* [12]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [17].

Let X be a vector space, $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t) x + t y,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t) x + t y],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [2, p. 2], [3, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t) x + t y] dt \le \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

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Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then we have the following norm inequality from (1.2) (see [16, p. 106])

$$\left\| \frac{x+y}{2} \right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}, \tag{1.3}$$

for any $x, y \in X$.

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator convex functions. The operator quasilinearity of some associated functionals are also provided.

In order to do that we need the following preliminary definitions and results.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [7, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [7] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function$f(t) = e^t$ is neither operator convex nor operator monotone.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [7] and the references therein. For other results, see [15], [9], [14] and [11]. For recent results, see [4], [5] and [6].

2. Some Hermite-Hadamard's Type Inequalities

We start with the following result:

Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality*

$$\left(f\left(\frac{A+B}{2}\right) \leq\right) \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \leq \int_0^1 f\left((1-t)A + tB\right)dt \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right]\left(\leq \frac{f(A)+f(B)}{2}\right). \tag{2.1}$$

Proof. First of all, since the function f is continous, the operator valued integral $\int_0^1 f\left((1-t)A + tB\right)dt$ exists for any selfadjoint operators A and B with spectra in I.

We give here two proofs, the first using only the definition of operator convex functions and the second using the classical Hermite-Hadamard inequality for real valued functions.

1. By the definition of operator convex functions we have the double inequality:

$$f\left(\frac{C+D}{2}\right) \leq \frac{1}{2}\left[f\left((1-t)C + tD\right) + f\left((1-t)D + tC\right)\right] \leq \frac{1}{2}\left[f(C) + f(D)\right] \tag{2.2}$$

for any $t \in [0,1]$ and any selfadjoint operators C and D with the spectra in I.

Integrating the inequality (2.2) over $t \in [0,1]$ and taking into account that

$$\int_0^1 f\left((1-t)C + tD\right)dt = \int_0^1 f\left((1-t)D + tC\right)dt$$

then we deduce the Hermite-Hadamard inequality for operator convex functions

$$f\left(\frac{C+D}{2}\right) \leq \int_0^1 f\left((1-t)C + tD\right)dt \leq \frac{1}{2}\left[f(C) + f(D)\right] \tag{HHO}$$

that holds for any selfadjoint operators C and D with the spectra in I.

Now, on making use of the change of variable $u = 2t$ we have

$$\int_0^{1/2} f\left((1-t)A + tB\right)dt = \frac{1}{2}\int_0^1 f\left((1-u)A + u\frac{A+B}{2}\right)du$$

and by the change of variable $u = 2t - 1$ we have

$$\int_{1/2}^1 f\left((1-t)A + tB\right)dt = \frac{1}{2}\int_0^1 f\left((1-u)\frac{A+B}{2} + uB\right)du.$$

Utilising the Hermite-Hadamard inequality (HHO) we can write

$$f\left(\frac{3A+B}{4}\right)\leq\int_0^1 f\left((1-u)A+u\frac{A+B}{2}\right)du$$
$$\leq\frac{1}{2}\left[f(A)+f\left(\frac{A+B}{2}\right)\right]$$

and

$$f\left(\frac{A+3B}{4}\right)\leq\int_0^1 f\left((1-u)\frac{A+B}{2}+uB\right)du$$
$$\leq\frac{1}{2}\left[f(A)+f\left(\frac{A+B}{2}\right)\right],$$

which by sumation and division by two produces the desired result (2.1).

2. Consider now $x\in H,\|x\|=1$ and two selfadjoint operators A and B with spectra in I. Define the real-valued function $\varphi_{x,A,B}:[0,1]\to\mathbb{R}$ given by $\varphi_{x,A,B}(t)=\langle f((1-t)A+tB)x,x\rangle$.

Since f is operator convex, then for any $t_1,t_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\varphi_{x,A,B}(\alpha t_1+\beta t_2)&=\langle f((1-(\alpha t_1+\beta t_2))A+(\alpha t_1+\beta t_2)B)x,x\rangle\\&=\langle f(\alpha[(1-t_1)A+t_1B]+\beta[(1-t_2)A+t_2B])x,x\rangle\\&\leq\alpha\langle f([(1-t_1)A+t_1B])x,x\rangle+\beta\langle f([(1-t_2)A+t_2B])x,x\rangle\\&=\alpha\varphi_{x,A,B}(t_1)+\beta\varphi_{x,A,B}(t_2)\end{aligned}$$

showing that $\varphi_{x,A,B}$ is a convex function on $[0,1]$.

Now we use the Hermite-Hadamard inequality for real-valued convex functions

$$g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b g(s)\,ds\leq\frac{g(a)+g(b)}{2}$$

to get that

$$\varphi_{x,A,B}\left(\frac{1}{4}\right)\leq 2\int_0^{1/2}\varphi_{x,A,B}(t)\,dt\leq\frac{\varphi_{x,A,B}(0)+\varphi_{x,A,B}\left(\frac{1}{2}\right)}{2}$$

and

$$\varphi_{x,A,B}\left(\frac{3}{4}\right)\leq 2\int_{1/2}^1\varphi_{x,A,B}(t)\,dt\leq\frac{\varphi_{x,A,B}\left(\frac{1}{2}\right)+\varphi_{x,A,B}(1)}{2}$$

which, by sumation and division by two, produce

$$\begin{aligned}&\frac{1}{2}\left\langle\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]x,x\right\rangle\\&\qquad\leq\int_0^1\langle f((1-t)A+tB)x,x\rangle\,dt\\&\qquad\qquad\leq\frac{1}{2}\left\langle\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]x,x\right\rangle.\end{aligned}\tag{2.3}$$

Finally, since by the continuity of the function f we have

$$\int_0^1\langle f((1-t)A+tB)x,x\rangle\,dt=\left\langle\int_0^1 f((1-t)A+tB)\,dtx,x\right\rangle$$

for any $x \in H, \|x\| = 1$ and any two selfadjoint operators A and B with spectra in I, we deduce from (2.3) the desired result (2.1). □

A simple consequence of the above theorem is that the integral is closer to the left bound than to the right, namely we can state:

Corollary 1. *With the assumptions in Theorem 1 we have the inequality*

$$(0 \leq) \int_0^1 f\left((1-t)A + tB\right) dt - f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2} - \int_0^1 f\left((1-t)A+tB\right) dt. \tag{2.4}$$

Remark 1. *Utilising different examples of operator convex or concave functions, we can provide inequalities of interest.*

If $r \in [-1,0] \cup [1,2]$ then we have the inequalities for powers of operators

$$\left(\left(\frac{A+B}{2}\right)^r \leq\right) \frac{1}{2}\left[\left(\frac{3A+B}{4}\right)^r + \left(\frac{A+3B}{4}\right)^r\right] \leq \int_0^1 \left((1-t)A+tB\right)^r dt \leq \frac{1}{2}\left[\left(\frac{A+B}{2}\right)^r + \frac{A^r+B^r}{2}\right] \left(\leq \frac{A^r+B^r}{2}\right) \tag{2.5}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$.

If $r \in (0,1)$ the inequalities in (2.5) hold with " $\geq$ " instead of " $\leq$ ".

We also have the following inequalities for logarithm

$$\left(\ln\left(\frac{A+B}{2}\right) \geq\right) \frac{1}{2}\left[\ln\left(\frac{3A+B}{4}\right) + \ln\left(\frac{A+3B}{4}\right)\right] \geq \int_0^1 \ln\left((1-t)A+tB\right) dt \geq \frac{1}{2}\left[\ln\left(\frac{A+B}{2}\right) + \frac{\ln(A)+\ln(B)}{2}\right] \left(\geq \frac{\ln(A)+\ln(B)}{2}\right) \tag{2.6}$$

for any two selfadjoint operators A and B with spectra in $(0,\infty)$.

3. Some Operator Quasilinearity Properties

Consider an operator convex function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval I and two distinct selfadjoint operators A, B with the spectra in I. We denote by $[A,B]$ the closed operator segment defined by the family of operators $\{(1-t)A + tB,\ t \in [0,1]\}$. We also define the operator-valued functional

$$\Delta_f(A,B;t) := (1-t) f(A) + t f(B) - f\left((1-t)A+tB\right) \geq 0 \tag{3.1}$$

in the operator order, for any $t \in [0,1]$.

The following result concerning an operator quasilinearity property for the functional $\Delta_f(\cdot,\cdot;t)$ may be stated:

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A, B two distinct selfadjoint operators with the spectra in I and $C \in [A, B]$ we have*

$$(0 \leq) \Delta_f (A, C; t) + \Delta_f (C, B; t) \leq \Delta_f (A, B; t) \tag{3.2}$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_f (\cdot, \cdot; t)$ is operator superadditive as a function of interval.

If $[C, D] \subset [A, B]$, then

$$(0 \leq) \Delta_f (C, D; t) \leq \Delta_f (A, B; t) \tag{3.3}$$

for each $t \in [0, 1]$, i.e., the functional $\Delta_f (\cdot, \cdot; t)$ is operator nondecreasing as a function of interval.

Proof. Let $C = (1 - s) A + sB$ with $s \in (0, 1)$. For $t \in (0, 1)$ we have

$$\Delta_f (C, B; t) = (1 - t) f ((1 - s) A + sB) + tf (B) \\ - f ((1 - t) [(1 - s) A + sB] + tB)$$

and

$$\Delta_f (A, C; t) = (1 - t) f (A) + tf ((1 - s) A + sB) \\ - f ((1 - t) A + t [(1 - s) A + sB])$$

giving that

$$\begin{aligned} &\Delta_f (A, C; t) + \Delta_f (C, B; t) - \Delta_f (A, B; t) \\ &= f ((1 - s) A + sB) + f ((1 - t) A + tB) \\ &- f ((1 - t) (1 - s) A + [(1 - t) s + t] B) - f ((1 - ts) A + tsB). \end{aligned} \tag{3.4}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$\frac{\varphi (t_1) - \varphi (t_2)}{t_1 - t_2} \leq \frac{\varphi (s_1) - \varphi (s_2)}{s_1 - s_2}. \tag{3.5}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi (t) := \frac{\varphi (t) - \varphi (a)}{t - a}$$

is monotonic nondecreasing on $I \setminus \{a\}$. Utilising this property repeatedly we have

$$\begin{aligned} \frac{\varphi (t_1) - \varphi (t_2)}{t_1 - t_2} &\leq \frac{\varphi (s_1) - \varphi (t_2)}{s_1 - t_2} = \frac{\varphi (t_2) - \varphi (s_1)}{t_2 - s_1} \\ &\leq \frac{\varphi (s_2) - \varphi (s_1)}{s_2 - s_1} = \frac{\varphi (s_1) - \varphi (s_2)}{s_1 - s_2}, \end{aligned}$$

which proves the inequality (3.5).

For a vector $x \in H$, with $\|x\| = 1$, consider the function $\varphi_x : [0, 1] \to \mathbb{R}$ given by $\varphi_x (t) := \langle f ((1 - t) A + tB) x, x\rangle$. Since f is operator convex on I it follows that φ_x is convex on $[0, 1]$. Now, if we consider, for given $t, s \in (0, 1)$,

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1 - t) s =: s_2,$$

then $\varphi_x(t_1) = \langle f((1-ts)A+tsB)x,x\rangle$ and $\varphi_x(t_2) = \langle f((1-t)A+tB)x,x\rangle$ giving that

$$\frac{\varphi_x(t_1)-\varphi_x(t_2)}{t_1-t_2} = \left\langle \left[\frac{f((1-ts)A+tsB)-f((1-t)A+tB)}{t(s-1)}\right]x,x\right\rangle .$$

Also

$$\varphi_x(s_1) = \langle f((1-s)A+sB)x,x\rangle$$

and

$$\varphi_x(s_2) = \langle f((1-t)(1-s)A+[(1-t)s+t]B)x,x\rangle$$

giving that

$$\begin{aligned}&\frac{\varphi_x(s_1)-\varphi_x(s_2)}{s_1-s_2}\\ &= \left\langle \frac{f((1-s)A+sB)-f((1-t)(1-s)A+[(1-t)s+t]B)}{t(s-1)}x,x\right\rangle .\end{aligned}$$

Utilising the inequality (3.5) and multiplying with $t(s-1)<0$ we deduce the following inequality in the operator order

$$\begin{aligned}&f((1-ts)A+tsB)-f((1-t)A+tB)\\ &\geq f((1-s)A+sB)-f((1-t)(1-s)A+[(1-t)s+t]B).\end{aligned} \tag{3.6}$$

Finally, by (3.4) and (3.6) we get the desired result (3.2).

Applying repeatedly the superadditivity property we have for $[C,D]\subset[A,B]$ that

$$\Delta_f(A,C;t)+\Delta_f(C,D;t)+\Delta_f(D,B;t)\leq\Delta_f(A,B;t)$$

giving that

$$0\leq\Delta_f(A,C;t)+\Delta_f(D,B;t)\leq\Delta_f(A,B;t)-\Delta_f(C,D;t)$$

which proves (3.3). □

For $t=\frac{1}{2}$ we consider the functional

$$\Delta_f(A,B):=\Delta_f\left(A,B;\frac{1}{2}\right)=\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Delta_f(\cdot,\cdot;t)$. We are able then to state the following

Corollary 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an operator convex function on the interval I. Then for each A,B two distinct selfadjoint operators with the spectra in I we have the following bounds in the operator order*

$$\inf_{C\in[A,B]}\left[f\left(\frac{A+C}{2}\right)+f\left(\frac{C+B}{2}\right)-f(C)\right]=f\left(\frac{A+B}{2}\right) \tag{3.7}$$

and

$$\sup_{C,D\in[A,B]}\left[\frac{f(C)+f(D)}{2}-f\left(\frac{C+D}{2}\right)\right]=\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right). \tag{3.8}$$

Proof. By the superadditivity of the functional $\Delta_f(\cdot,\cdot)$ we have for each $C\in[A,B]$ that

$$\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\geq\frac{f(A)+f(C)}{2}-f\left(\frac{A+C}{2}\right)+\frac{f(C)+f(B)}{2}-f\left(\frac{C+B}{2}\right)$$

which is equivalent with

$$f\left(\frac{A+C}{2}\right)+f\left(\frac{C+B}{2}\right)-f(C)\geq f\left(\frac{A+B}{2}\right). \tag{3.9}$$

Since the equality case in (3.9) is realized for either $C=A$ or $C=B$ we get the desired bound (3.7).

The bound (3.8) is obvious by the monotonicity of the functional $\Delta_f(\cdot,\cdot)$ as a function of interval. □

Consider now the following functional

$$\Gamma_f(A,B;t):=f(A)+f(B)-f((1-t)A+tB)-f((1-t)B+tA),$$

where, as above, $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I and A,B are selfadjoint operators with the spectra in I while $t\in[0,1]$.

We notice that

$$\Gamma_f(A,B;t)=\Gamma_f(B,A;t)=\Gamma_f(A,B;1-t)$$

and

$$\Gamma_f(A,B;t)=\Delta_f(A,B;t)+\Delta_f(A,B;1-t)\geq 0$$

for any A,B selfadjoint operators with the spectra in I and $t\in[0,1]$.

Therefore, we can state the following result as well

Corollary 3. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an operator convex function on the interval I. Then for each A,B two distinct selfadjoint operators with the spectra in I, the functional $\Gamma_f(\cdot,\cdot;t)$ is operator superadditive and operator nondecreasing as a function of interval.*

In particular, if $C\in[A,B]$ then we have the inequality

$$\begin{aligned}\frac{1}{2}[f((1-t)A+tB)+f((1-t)B+tA)]&\leq\frac{1}{2}[f((1-t)A+tC)+f((1-t)C+tA)]\\&\quad+\frac{1}{2}[f((1-t)C+tB)+f((1-t)B+tC)]-f(C).\end{aligned}\tag{3.10}$$

Also, if $C,D\in[A,B]$ then we have the inequality

$$\begin{aligned}f(A)+f(B)&-f((1-t)A+tB)-f((1-t)B+tA)\\&\geq f(C)+f(D)-f((1-t)C+tD)-f((1-t)C+tD)\end{aligned}\tag{3.11}$$

for any $t\in[0,1]$.

Perhaps the most interesting functional we can consider is the following one:

$$\Theta_f(A,B)=\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt. \tag{3.12}$$

Notice that, by the second Hermite-Hadamard inequality for operator convex functions we have that $\Theta_f(A,B) \geq 0$ in the operator order.

We also observe that

$$\Theta_f(A,B) = \int_0^1 \Delta_f(A,B;t)\,dt = \int_0^1 \Delta_f(A,B;1-t)\,dt. \tag{3.13}$$

Utilising this representation, we can state the following result as well:

Corollary 4. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for each A,B two distinct selfadjoint operators with the spectra in I, the functional $\Theta_f(\cdot,\cdot)$ is operator superadditive and operator nondecreasing as a function of interval. Moreover, we have the bounds in the operator order*

$$\inf_{C\in[A,B]}\left[\int_0^1 [f((1-t)A+tC)+f((1-t)C+tB)]\,dt - f(C)\right] = \int_0^1 f((1-t)A+tB)\,dt \tag{3.14}$$

and

$$\sup_{C,D\in[A,B]}\left[\frac{f(C)+f(D)}{2} - \int_0^1 f((1-t)C+tD)\,dt\right] = \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A+tB)\,dt. \tag{3.15}$$

Remark 2. *The above inequalities can be applied to various concrete operator convex function of interest.*

If we choose, for instance, the inequality (3.15), then we get the following bounds in the operator order

$$\sup_{C,D\in[A,B]}\left[\frac{C^r+D^r}{2} - \int_0^1 ((1-t)C+tD)^r\,dt\right] = \frac{A^r+B^r}{2} - \int_0^1 ((1-t)A+tB)^r\,dt, \tag{3.16}$$

where $r \in [-1,0]\cup[1,2]$ and A,B are selfadjoint operators with spectra in $(0,\infty)$.

If $r \in (0,1)$ then

$$\sup_{C,D\in[A,B]}\left[\int_0^1 ((1-t)C+tD)^r\,dt - \frac{C^r+D^r}{2}\right] = \int_0^1 ((1-t)A+tB)^r\,dt - \frac{A^r+B^r}{2}, \tag{3.17}$$

and A,B are selfadjoint operators with spectra in $(0,\infty)$.

We also have the operator bound for the logarithm

$$\sup_{C,D\in[A,B]}\left[\int_0^1 \ln((1-t)C+tD)\,dt - \frac{\ln(C)+\ln(D)}{2}\right] = \int_0^1 \ln((1-t)A+tB)\,dt - \frac{\ln(A)+\ln(B)}{2}, \tag{3.18}$$

where A,B are selfadjoint operators with spectra in $(0,\infty)$.

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