ON SOME HADAMARD TYPE INEQUALITIES FOR PRODUCT OF DIFFERENT KINDS OF CONVEX FUNCTIONS

MEVLUT TUNC

Abstract. In this paper, some inequalities Hadamard-type for product of different kinds of convex functions are given.

1. Introduction

The following definition is well known in the literature: a function $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Definition 1. *[1] A function $f : [0,b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have :*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{1.2}$$

Remark 1. *[2] For $m = 1$, we recapture the concept of convex functions defined on $[0,b]$ and for $m = 0$ we get the concept of starshaped functions $[0,b]$ on . We recall that $f : [0,b] \to \mathbb{R}$ is starshaped if*

$$f(tx) \leq tf(x) \tag{1.3}$$

for all $t \in [0,1]$ and $x \in [0,b]$.

Definition 2. *[1] A function $f : [0,b] \to \mathbb{R}$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0,1]^2$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have :*

$$f(tx + m(1-t)y) \leq t^{\alpha}f(x) + m(1-t^{\alpha})f(y) \tag{1.4}$$

Note that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: increasing, $\alpha-$starshaped, starshaped, $m-$convex, convex and $\alpha-$convex.

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Definition 3. *[9] We say that $f : I \to \mathbb{R}$ is Gudunova- Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t} \tag{1.5}$$

Definition 4. *[3] Let $s \in (0,1]$. A function $f : (0,\infty] \to [0,\infty]$ is said to be $s-$convex (in the second sense) if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.6}$$

for all $x, y \in (0,b]$ and $t \in [0,1]$.

In 1978, Breckner introduced $s-$convex functions as a generalization of convex functions [14]. Also, in that one work Breckner proved the important fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function [15]. A number of properties and connections with $s-$convexity in the first sense are discussed in paper [10]. Of course, $s-$convexity means just convexity when $s = 1$.

Definition 5. *[4] We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P-$ function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq f(x) + f(y) \tag{1.7}$$

Definition 6. *[5] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y) \tag{1.8}$$

If inequality (1.8) is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$. Obviously, if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

Theorem 1. *[6] Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense, where $s \in (0,1)$, and let $a, b \in [0,\infty), a < b$. If $f \in L_1([a,b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

.

In [3], Pachpatte established two new Hadamard-type inequalities for products of convex functions. An analogous result for $s-$convex functions is due to Kirmaci at. al.[7].

Theorem 2. *[3] Let $f,g:[a,b]\rightarrow[0,\infty)$ be convex functions on $[a,b]\subset\mathbb{R}$, $a<b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{1.10}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.11}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

Theorem 3. *[7] Let $f,g:[a,b]\rightarrow\mathbb{R}$, $a,b\in[0,\infty)$,$a<b$ be functions such that g and fg are in $L_1([a,b])$. If f is convex and nonnegative on $[a,b]$, and if g is $s-$convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \tag{1.12}$$
$$\leq\frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b)$$

In [4], Dragomir et. al. proved two inequalities of Hadamard type for classes of Godunova-Levin functions and $P-$functions.

Theorem 4. *[4] Let $f\in Q(I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{4}{b-a}\int_a^b f(x)\,dx \tag{1.13}$$

Theorem 5. *[4] Let $f\in P(I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2}{b-a}\int_a^b f(x)\,dx\leq2\left[f(a)+f(b)\right] \tag{1.14}$$

.

The main purpose of this paper is to establish new inequalities like those given in the above theorems, but now we do for product of different kinds class of convex functions.

2. **Main Results**

We shall make use of the following theorems.

Theorem 6. *Let $h:J\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be a positive function, $a,b\in[0,\infty)$,$a<b$ be such that $f,g:[a,b]\rightarrow\mathbb{R}$ functions and $fg\in L_1([a,b])$, $h\in L_1([0,1])$. If f, $h-$convex and $g\in K_s^2([a,b])$ for $s,t\in[0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\leq M(a,b)\int_0^1 h(t)\,t^s dt+N(a,b)\int_0^1 h(1-t)\,t^s dt \tag{2.1}$$

where $M(a,b)=f(a)\,g(a)+f(b)\,g(b)$ and $N(a,b)=f(a)\,g(b)+f(b)\,g(a)$.

Proof. Since f, $h-$convex and $g \in K_s^2([a,b])$, on $[a,b]$ for $x = ta + (1-t)b$,

$$\begin{aligned} f(ta+(1-t)b) &\leq h(t)f(a) + h(1-t)f(b) \\ g(ta+(1-t)b) &\leq t^s g(a) + (1-t)^s g(b) \end{aligned}$$

for all $t \in [0,1]$. Since f and g are nonnegative, so

$$\begin{aligned} & f(ta+(1-t)b)\, g(ta+(1-t)b) \\ \leq\ & [h(t)f(a) + h(1-t)f(b)]\,[t^s g(a) + (1-t)^s g(b)] \\ =\ & h(t)t^s f(a)g(a) + h(t)(1-t)^s f(a)g(b) \\ & + h(1-t)t^s f(b)g(a) + h(1-t)(1-t)^s f(b)g(b) \end{aligned}$$

we can write. Integrating both sides of the that inequality over $[0,1]$, we obtain

$$\begin{aligned} & f(ta+(1-t)b)\, g(ta+(1-t)b) \\ =\ & \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \\ \leq\ & f(a)g(a)\int_0^1 h(t)t^s dt + f(b)g(b)\int_0^1 h(1-t)(1-t)^s\,dt \\ & + f(b)g(a)\int_0^1 h(1-t)t^s dt + f(a)g(b)\int_0^1 h(t)(1-t)^s\,dt. \end{aligned}$$

and

$$\int_0^1 h(t)t^s dt = \int_0^1 h(1-t)(1-t)^s\,dt \text{ and } \int_0^1 h(1-t)t^s dt = \int_0^1 h(t)(1-t)^s\,dt$$

when above equalities are taken into account

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq\ & [f(a)g(a) + f(b)g(b)]\int_0^1 h(t)t^s dt \\ & + [f(b)g(a) + f(a)g(b)]\int_0^1 h(1-t)t^s dt \end{aligned}$$

the proof is complete. □

Remark 2. *In Theorem 6., if we choose $h(t) = t$ and $f(x) = 1$, where $s \in [0,1]$,then (2.1) reduces to right side of (1.9). So, we obtain*

$$\frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{g(a)+g(b)}{s+1}$$

Theorem 7. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $a, b \in [0,\infty)$,$a < b$ be such that $f, g : [a,b] \to \mathbb{R}$ functions and $fg \in L_1([a,b])$, $h \in L_1([0,1])$. If f, $h-$convex and $g \in K_s^2([a,b])$ for $s, t \in [0,1]$, then*

$$\begin{aligned} & \frac{2^{s-1}}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \qquad (2.2) \\ \leq\ & M(a,b)\int_0^1 h(1-t)t^s dt + N(a,b)\int_0^1 h(t)t^s dt \end{aligned}$$

Proof. Since f, $h-$convex and $g\in K_s^2\left([a,b]\right)$, on $[a,b]$ for $\frac{a+b}{2}=\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}$, so we can write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq h\left(\frac{1}{2}\right)\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] \\ g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2^s}\left[g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right] \end{aligned}$$

Since and are nonnegative, so we can write

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] \\ &\quad \text{x}\left[g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right] \end{aligned}$$

$$\begin{aligned} &= h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)g\left((1-t)a+tb\right)+f\left((1-t)a+tb\right)g\left(ta+(1-t)b\right)\right] \\ &\leq h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[h(t)f(a)+h(1-t)f(b)\right]\left[(1-t)^s g(a)+t^s g(b)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[h(1-t)f(a)+h(t)f(b)\right]\left[t^s g(a)+(1-t)^s g(b)\right] \\ &= h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[\left[h(t)(1-t)^s+h(1-t)t^s\right]f(a)g(a)+\left[h(t)t^s+h(1-t)(1-t)^s\right]f(a)g(b)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[\left[h(1-t)(1-t)^s+h(t)t^s\right]f(b)g(a)+\left[h(1-t)t^s+h(t)(1-t)^s\right]f(b)g(b)\right] \\ &= h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[h(t)(1-t)^s+h(1-t)t^s\right]\left[f(a)g(a)+f(b)g(b)\right] \\ &\quad +h\left(\frac{1}{2}\right)\frac{1}{2^s}\left[h(t)t^s+h(1-t)(1-t)^s\right]\left[f(a)g(b)+f(b)g(a)\right] \end{aligned}$$

Thus, we get

$$\begin{aligned} &\frac{2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)}-\left[f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right] \\ &\leq \left[h(t)(1-t)^s+h(1-t)t^s\right]M(a,b)+\left[h(t)t^s+h(1-t)(1-t)^s\right]N(a,b) \end{aligned}$$

Integrating both sides of that inequality over , we obtain;

$$\begin{aligned} & \frac{2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)}{h\left(\frac{1}{2}\right)} - \frac{2}{b-a}\int_a^b f(x)\, g(x)\, dx \\ \leq\ & M(a,b)\left\{\int_0^1 h(t)(1-t)^s dt + \int_0^1 h(1-t)\, t^s dt\right\} \\ & + N(a,b)\left\{\int_0^1 h(t)\, t^s dt + \int_0^1 h(1-t)(1-t)^s dt\right\} \end{aligned}$$

and

$$\int_0^1 h(t)\, t^s dt = \int_0^1 h(1-t)(1-t)^s dt \text{ and } \int_0^1 h(1-t)\, t^s dt = \int_0^1 h(t)(1-t)^s dt$$

When above equalities are taken into account, so the proof is complete. □

Remark 3. *In Theorem 7, if we choose $h(t) = t$ and $f(x) = 1$, where $s \in [0,1]$, then (2.2) reduces to (1.12). So, we obtain*

$$\begin{aligned} \frac{2^{s-1} f\left(\frac{a+b}{2}\right)}{1/2} - \frac{1}{b-a}\int_a^b f(x)\, dx &\leq M(a,b)\int_0^1 (1-t)\, t^s dt + N(a,b)\int_0^1 t t^s dt \\ &= M(a,b)\int_0^1 \left(t^s - t^{s+1}\right) dt + N(a,b)\int_0^1 t^{s+1} dt \\ &= \frac{M(a,b)}{(s+1)(s+2)} + \frac{N(a,b)}{s+2} \end{aligned}$$

Theorem 8. *Let $a, b \in [0,\infty)$, $a < b$ such that $f, g : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing and $fg \in L_1([a,b])$. If f, m_1-convex , g, $(\alpha, m_2)-$convex on $[a,b]$ for $m_1 \in [0,1]$ and $(\alpha, m_2) \in [0,1]^2$, then*

$$\begin{aligned} & \frac{1}{m_1 b - a}\int_a^{m_1 b} f(x)\, dx \frac{1}{m_2 b - a}\int_a^{m_2 b} g(x)\, dx \\ \leq\ & Ef(a)\, g(a) + Lf(a)\, g(b) + If(b)\, g(a) + Ff(b)\, g(b) \end{aligned} \tag{2.3}$$

where $E = \frac{1}{\alpha+2}$, $L = \frac{m_2\alpha}{2(\alpha+2)}$, $I = \frac{m_1}{(\alpha+1)(\alpha+2)}$, $F = \frac{m_1 m_2 \alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}$.

Proof. If f, m_1-convex , g, $(\alpha, m_2)-$convex on $[a,b]$ for $m_1 \in [0,1]$ and $(\alpha, m_2) \in [0,1]^2$, for $x = ta + (1-t)\, b$, $t \in [0,1]$, so

$$\begin{aligned} f(ta + m_1(1-t)\, b) &\leq tf(a) + m_1(1-t)\, f(b), \\ g(ta + m_2(1-t)\, b) &\leq t^\alpha g(a) + m_2(1-t^\alpha)\, g(b) \end{aligned}$$

since f and g are nonnegative we write that

$$\begin{aligned} & f(ta + m_1(1-t)\, b)\, g(ta + m_2(1-t)\, b) \\ \leq\ & f(a)\, g(a)\, t^{\alpha+1} + m_2 f(a)\, g(b)\, t(1-t^\alpha) \\ & + m_1 f(b)\, g(a)\, t^\alpha(1-t) + m_1 m_2 f(b)\, g(b)(1-t)(1-t^\alpha) \end{aligned}$$

Integrating both sides of that inequality over $[0,1]$,by using Chebychev inequality

$$\frac{1}{m_1 b - a}\int_a^{m_1 b} f(x)\, dx \frac{1}{m_2 b - a}\int_a^{m_2 b} g(x)\, dx$$

$$\begin{aligned}
&\leq \int_0^1 f(ta+m_1(1-t)b)\,dt \int_0^1 g(ta+m_2(1-t)b)\,dt \\
&\leq \int_0^1 f(ta+m_1(1-t)b)\,g(ta+m_2(1-t)b)\,dt \\
&\leq f(a)g(a)\int_0^1 t^{\alpha+1}dt + m_2 f(a)g(b)\int_0^1 t(1-t^{\alpha})\,dt \\
&\quad +m_1 f(b)g(a)\int_0^1 t^{\alpha}(1-t)\,dt + m_1m_2 f(b)g(b)\int_0^1 (1-t)(1-t^{\alpha})\,dt \\
&= \frac{1}{\alpha+2}f(a)g(a)+\frac{m_2\alpha}{2(\alpha+2)}f(a)g(b) \\
&\quad +\frac{m_1}{(\alpha+1)(\alpha+2)}f(b)g(a)+\frac{m_1m_2\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}f(b)g(b)
\end{aligned}$$

We can write that the proof is complete. □

Theorem 9. *Let $a,b\in[0,\infty)$,$a<b$ such that $f,g:[a,b]\to\mathbb{R}$ be integrable functions, both increasing or both decreasing and $fg\in L_1([a,b])$. If f,g $(\alpha,m)-$convex on $[a,b]$ for $(\alpha,m)\in[0,1]^2$, then*

$$\begin{aligned}
&\frac{1}{m_1b-a}\int_a^{m_1b} f(x)\,dx\frac{1}{m_2b-a}\int_a^{m_2b} g(x)\,dx \qquad (2.4)\\
&\leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \\
&\leq Sf(a)g(a)+Ef(a)g(b)+Mf(b)g(a)+Af(b)g(b)
\end{aligned}$$

where $S=\frac{1}{\alpha_1+\alpha_2+1}$, $E=\frac{m_2\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)}$, $M=\frac{m_1\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)}$, $A=\frac{m_1m_2\alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)}$.

Proof. Since f,g $(\alpha,m)-$convex and both increasing or both decreasing functions,

$$\begin{aligned}
f(ta+m_1(1-t)b) &\leq t^{\alpha_1}f(a)+m_1(1-t^{\alpha_1})f(b), \\
g(ta+m_2(1-t)b) &\leq t^{\alpha_2}g(a)+m_2(1-t^{\alpha_2})g(b)
\end{aligned}$$

then two inequalities above are multiplied on either side (i.e. from left to left and right to right)

$$\begin{aligned}
&f(ta+m_1(1-t)b)\,g(ta+m_2(1-t)b) \\
&\leq [t^{\alpha_1}f(a)+m_1(1-t^{\alpha_1})f(b)]\,[t^{\alpha_2}g(a)+m_2(1-t^{\alpha_2})g(b)] \\
&= t^{\alpha_1}t^{\alpha_2}f(a)g(a)+m_2t^{\alpha_1}(1-t^{\alpha_2})f(a)g(b) \\
&\quad +m_1t^{\alpha_2}(1-t^{\alpha_1})f(b)g(a)+m_1m_2(1-t^{\alpha_1})(1-t^{\alpha_2})f(b)g(b)
\end{aligned}$$

[13] In Classical and New Inequalities in Analysis (pp239); Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions, both increasing or both decreasing.

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \geq \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx \qquad (2.5)$$

On the other hand, if one of the functions f or g is nonincreasing and the other nondecreasing, then the inequality in (2.5) is reversed. By using Chebyshev inequality and generalization Szegö and Weinberger, we can write

$$\begin{aligned}
&\int_0^1 f(ta+m_1(1-t)b)\,g(ta+m_2(1-t)b)\,dt \\
\geq\ &\int_0^1 f(ta+m_1(1-t)b)\,dt\int_0^1 g(ta+m_2(1-t)b)\,dt \\
=\ &\frac{1}{m_1b-a}\int_a^{m_1b} f(x)\,dx\,\frac{1}{m_2b-a}\int_a^{m_2b} g(x)\,dx
\end{aligned}$$

Integrating both sides of the product over $[0,1]$

$$\begin{aligned}
&\frac{1}{m_1b-a}\int_a^{m_1b} f(x)\,dx\,\frac{1}{m_2b-a}\int_a^{m_2b} g(x)\,dx \\
\leq\ & f(a)\,g(a)\int_0^1 t^{\alpha_1+\alpha_2}dt+m_2f(a)\,g(b)\int_0^1 t^{\alpha_1}\left(1-t^{\alpha_2}\right)dt \\
&+m_1f(b)\,g(a)\int_0^1 t^{\alpha_2}\left(1-t^{\alpha_1}\right)dt \\
&+m_1m_2f(b)\,g(b)\int_0^1\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right)dt \\
=\ &\frac{f(a)\,g(a)}{\alpha_1+\alpha_2+1}+f(a)\,g(b)\,\frac{m_2\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)} \\
&+f(b)\,g(a)\,\frac{m_1\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)} \\
&+f(b)\,g(b)\,\frac{m_1m_2\alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)}
\end{aligned}$$

We can write that the proof is complete. □

Remark 4. *In Theorem 9, if we particularly choose $m_1=m_2=\alpha_1=\alpha_2=1$ then (2.4) reduces to (1.10).*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)$$

In addition, if we choose $g(x)=1$, then we have the right side of Hermite-Hadamard inequality.

Theorem 10. *Let $a,b\in[0,\infty)$, $a<b$ such that $f,g:[a,b]\to\mathbb{R}$ be integrable functions, both increasing or both decreasing and $fg\in L_1([a,b])$. If f, m_1-convex , g, $(\alpha,m_2)-$convex on $[a,b]$ for $\alpha\in[0,1]$ and $m_{1,2}\in(0,1]$, then*

$$\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\leq\min\{E,L,I,F\} \tag{2.6}$$

where

$$\begin{aligned}
E\ =\ & f(a)\,g(a)\,\frac{1}{\alpha+2}+m_2f(a)\,g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1g(a)\,f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}$$

$$\begin{aligned}
L\ =\ & f(b)\,g(b)\,\frac{1}{\alpha+2}+m_2f(b)\,g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1g(b)\,f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}+m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}$$

$$
\begin{aligned}
I &= f(b)\,g(a)\frac{1}{\alpha+2}+m_2 f(b)\,g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(a)\,f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}+m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}
$$

$$
\begin{aligned}
F &= f(a)\,g(b)\frac{1}{\alpha+2}+m_2 f(a)\,g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(b)\,f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)}+m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}
$$

Proof. Since f, m_1-convex , g, $(\alpha, m_2)-$convex on $[a,b]$

$$
\begin{aligned}
f(ta+(1-t)b) &= f\left(ta+m_1(1-t)\frac{b}{m_1}\right) \le tf(a)+m_1(1-t)\,f\left(\frac{b}{m_1}\right), \\
g(ta+(1-t)b) &= g\left(ta+m_2(1-t)\frac{b}{m_2}\right) \le t^{\alpha}g(a)+m_2(1-t^{\alpha})\,g\left(\frac{b}{m_2}\right)
\end{aligned}
$$

We can write, f and g are nonnegative for $\forall t \in [0,1]$

$$
\begin{aligned}
f(ta+(1-t)b)\,g(ta+(1-t)b) &\le f(a)\,g(a)\,t^{\alpha+1}+m_2 f(a)\,g\left(\frac{b}{m_2}\right)t(1-t^{\alpha}) \\
&+m_1 g(a)\,f\left(\frac{b}{m_1}\right)t^{\alpha}(1-t) \\
&+m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)(1-t)(1-t^{\alpha})
\end{aligned}
$$

Integrating both sides of the product over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt \\
\le\; & f(a)\,g(a)\int_0^1 t^{\alpha+1}dt+m_2 f(a)\,g\left(\frac{b}{m_2}\right)\int_0^1 t(1-t^{\alpha})\,dt \\
&+m_1 g(a)\,f\left(\frac{b}{m_1}\right)\int_0^1 t^{\alpha}(1-t)\,dt \\
&+m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 (1-t)(1-t^{\alpha})\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt \\
\le\; & f(a)\,g(a)\frac{1}{\alpha+2}+m_2 f(a)\,g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(a)\,f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)} \\
&+m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}
$$

similarly

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt \\
\leq\ & f(b)\,g(b)\,\frac{1}{\alpha+2}+m_2 f(b)\,g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(b)\,f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)} \\
&+m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt \\
\leq\ & f(b)\,g(a)\,\frac{1}{\alpha+2}+m_2 f(b)\,g\left(\frac{b}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(a)\,f\left(\frac{a}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)} \\
&+m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{b}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt \\
\leq\ & f(a)\,g(b)\,\frac{1}{\alpha+2}+m_2 f(a)\,g\left(\frac{a}{m_2}\right)\frac{\alpha}{2(\alpha+2)} \\
&+m_1 g(b)\,f\left(\frac{b}{m_1}\right)\frac{1}{(\alpha+1)(\alpha+2)} \\
&+m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{a}{m_2}\right)\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)}
\end{aligned}$$

we can write. By using ability Chebychev integral inequality on above four different inequalities, we obtain

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &\geq \frac{1}{b-a}\int_a^b f(x)\,dx\frac{1}{b-a}\int_a^b g(x)\,dx \\
&= \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx
\end{aligned}$$

That the proof is complete. □

Theorem 11. *Let $a,b\in[0,\infty)$, $a<b$ such that $f,g:[a,b]\to\mathbb{R}$ be integrable functions and $f,g,fg\in L_1([a,b])$. If f and g are bounded and positive functions, one of which is nondecreasing and the other nonincreasing, and f belongs to the class $SX(h,I)$, g belongs to the class of $Q(I)$, then*

$$\begin{aligned}
&\frac{1}{4h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \qquad (2.7)\\
<\ & \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx
\end{aligned}$$

Proof. Because of $f \in SX(h,I)$ and $g \in Q(I)$ and $\frac{a+b}{2} = \frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}$,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\quad \mathrm{x} g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\le 2h\left(\frac{1}{2}\right)\left[f(ta+(1-t)b) + f((1-t)a+tb)\right] \\ &\quad \mathrm{x}\left[g(ta+(1-t)b) + g((1-t)a+tb)\right] \\ &= 2h\left(\frac{1}{2}\right) f(ta+(1-t)b)\, g(ta+(1-t)b) \\ &\quad +2h\left(\frac{1}{2}\right) f((1-t)a+tb)\, g((1-t)a+tb) \\ &\quad +2h\left(\frac{1}{2}\right)\left[f(ta+(1-t)b)\, g((1-t)a+tb)\right] \\ &\quad +2h\left(\frac{1}{2}\right) f((1-t)a+tb)\, g(ta+(1-t)b) \end{aligned}$$

we write, integrating both sides of the product over $[0,1]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \le\ & 2h\left(\frac{1}{2}\right)\int_0^1 f(ta+(1-t)b)\, g(ta+(1-t)b)\, dt \\ & +2h\left(\frac{1}{2}\right)\int_0^1 f((1-t)a+tb)\, g((1-t)a+tb)\, dt \\ & +2h\left(\frac{1}{2}\right)\int_0^1 f(ta+(1-t)b)\, dt \int_0^1 g((1-t)a+tb)\, dt \\ & +2h\left(\frac{1}{2}\right)\int_0^1 f((1-t)a+tb)\, dt \int_0^1 g(ta+(1-t)b)\, dt \end{aligned}$$

In New and Classic Inequalities in Analysis [pp240] , [12],[13] Winckler, A.; If f and g are bounded and positive functions, one of which is nondecreasing and the other nonincreasing for $x > 0$, then

$$\int_0^x f(x)\, g(x)\, dx < \frac{1}{x}\int_0^x f(x)\, dx \int_0^x g(x)\, dx.$$

By using that type of Chebyshev inequality

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) < \frac{4h\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)\, g(x)\, dx + \frac{4h\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)\, dx \int_a^b g(x)\, dx$$

We can write that the proof is complete. □

Theorem 12. *Let $a,b \in [0,\infty)$,$a<b$, $I=[a,b]$ with $f,g : [a,b] \to \mathbb{R}$ be functions f,g and fg are in $L_1([a,b])$. If f is convex and g belongs to the class of $P(I)$ then,*

$$\frac{1}{b-a}\int_a^b f(x)\, g(x)\, dx \le \frac{M(a,b) + N(a,b)}{2} \tag{2.8}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f is convex and $g\in P(I)$,

$$\begin{aligned} f(ta+(1-t)b) &\leq tf(a)+(1-t)f(b) \\ g(ta+(1-t)b) &\leq g(a)+g(b) \end{aligned}$$

Since f and g are nonnegative, we obtain,

$$\begin{aligned} f(ta+(1-t)b)g(ta+(1-t)b) &\leq [tf(a)+(1-t)f(b)][g(a)+g(b)] \\ &= tf(a)g(a)+tf(a)g(b)+(1-t)f(b)g(a)+(1-t)f(b)g(b) \end{aligned}$$

integrating both sides of the product over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b) \\ \leq\ &[f(a)g(a)+f(a)g(b)]\int_0^1 tdt+[f(b)g(a)+f(b)g(b)]\int_0^1(1-t)dt \end{aligned}$$

The proof is complete. □

Theorem 13. *Let $a,b\in[0,\infty)$, $a<b, I=[a,b]$ and $f,g:[a,b]\to\mathbb{R}$ be functions and $f\in L_1([a,b])$. If $f\in Q(I)$ and $g\in P(I)$ then,*

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{8(g(a)+g(b))}{b-a}\int_a^b f(x)\,dx. \tag{2.9}$$

Proof. Since $f\in Q(I)$ and $g\in P(I)$ and $\frac{a+b}{2}=\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}$,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq 2(f(ta+(1-t)b)+f((1-t)a+tb)) \\ g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) \\ &\leq 2(g(a)+g(b)) \end{aligned}$$

we write. Since f and g are nonnegative, we obtain,

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq 4(g(a)+g(b))(f(ta+(1-t)b)+f((1-t)a+tb))$$

integrating both sides of the product over $[0,1]$, we obtain

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq\ &4(g(a)+g(b))\int_0^1(f(ta+(1-t)b)+f((1-t)a+tb))\,dt \\ =\ &\frac{8(g(a)+g(b))}{b-a}\int_a^b f(x)\,dx \end{aligned}$$

The proof is complete. □

3. Applications to Some Special Means

We now consider the applications of our Theorems to the following special means
a) The power mean:

$$M_p = M_p(x_1, ..., x_n) := \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{1/p}, \quad a, b \geq 0,$$

b) The arithmetic mean:

$$A = A(a,b) := \frac{a+b}{2}, \quad a, b \geq 0,$$

c) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \quad a, b \geq 0,$$

d) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a, b \geq 0,$$

e) The quadratic mean:

$$K = K(a,b) := \sqrt{\frac{a^2+b^2}{2}} \quad a, b \geq 0,$$

f) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if} \quad a \neq b \end{cases}, \quad a, b \geq 0,$$

g) The Identric mean.

$$I = I(a,b) := \begin{cases} a & \text{if} \quad a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad a \neq b \end{cases}, \quad a, b \geq 0,$$

h) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R}\setminus\{-1, 0\}; \quad a, b > 0.$$

The following inequality is well known in the literature:

$$H \leq G \leq L \leq I \leq A \leq K$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

The following propositions holds:

Proposition 1. *Let $a, b \in \mathbb{R}_+$, $0 \notin [a,b]$, and $n \in \mathbb{Z}$, $|n| \geq 2$. Then for all $s \in [0,1]$, we have*

$$\frac{L_n^n(a,b)}{2} \leq A(a^n, b^n)\frac{1}{s+1} + G^n(a,b)\frac{1}{(s+1)(s+2)} \tag{3.1}$$

Proof. The assertion follows from Theorem 6. applied for $f(x) = g(x) = x^{n/2}, x \in \mathbb{R}, n \in \mathbb{Z}, |n| \geq 2$ and $h(t) = t$. □

Proposition 2. *Let $a, b \in (0, \infty)$, $a < b$, and $n \in \mathbb{Z}$, $|n| \geq 2$. Then for all $s \in [0,1]$, we have*

$$\ln \frac{A^{2^s}(a,b)}{I(a,b)} \leq \frac{1}{(s+1)(s+2)} \ln G^2(a,b) + \frac{2}{s+2} G^2(\ln a, \ln b) \tag{3.2}$$

Proof. The proof is obvious from Theorem 7. applied $f(x) = g(x) = \sqrt{\ln x}$, $h(t) = t$, $x \in [0, \infty)$, $n \in \mathbb{Z}$, and $|n| \geq 2$ and $s \in [0,1]$. □

Proposition 3. *Let $a, b \in \mathbb{R}_+$, $0 \notin [a,b]$, and $n \in \mathbb{Z}$, $|n| \geq 2$. Then for all $p > 1$, we have*

$$L_n^n(a,b).M_p(a,b) - \frac{G^{n/2}(a,b)}{2(p+1)} \leq \frac{b-a}{2}\left(G^{n/2}(a,b) + M_{2n}^{2n}(a,b)\right) \tag{3.3}$$

Proof. The proff is immadiate from Theorem 12. applied for $f(x) = g(x) = x^n, x \in \mathbb{R}, n \in \mathbb{Z}$, and $|n| \geq 2$. □

Proposition 4. *Let $0 < a < b$. Then one has the inequality*

$$16A^2(a,b) \geq H(a,b)\,L(a,b) \tag{3.4}$$

Proof. If we choose in Theorem 13., $f(x) = g(x) = \frac{1}{x}$, we have that

$$\begin{aligned} \frac{2}{a+b}\frac{2}{a+b} &\leq 8\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{b-a}\int_a^b \frac{1}{x}dx \\ \frac{1}{A^2(a,b)} &\leq 16\left(\frac{a+b}{2ab}\right)\frac{1}{L(a,b)} \\ \frac{1}{A^2(a,b)} &\leq 16\frac{1}{H(a,b)}\frac{1}{L(a,b)} \end{aligned}$$

which is equivalent to (3.4). □

Proposition 5. *Let $0 < a < b < \infty$. Then one has the inequality*

$$\frac{3}{L^2(a,b)} \leq \frac{2K^2(a,b) + G^2(a,b)}{G^4(a,b)} \tag{3.5}$$

Proof. If we choose in Theorem 10., $f(x) = g(x) = \frac{1}{x}$ and $\alpha = m_1 = m_2 = 1$, we have that E;

$$\begin{aligned} \frac{1}{L^2(a,b)} &\leq \frac{1}{a^2}\frac{1}{\alpha+2} + m_2\frac{m_2}{ab}\frac{\alpha}{2(\alpha+2)} + m_1\frac{m_1}{ab}\frac{\alpha}{(\alpha+1)(\alpha+2)} \\ &\quad + m_1 m_2 \frac{m_1 m_2}{b^2}\frac{\alpha(\alpha+3)}{2(\alpha+1)(\alpha+2)} \\ &= \frac{1}{3a^2} + \frac{1}{3ab} + \frac{1}{3b^2} = \frac{1}{3}\left(\frac{a^2+b^2}{a^2b^2} + \frac{1}{ab}\right) \\ &= \frac{1}{3}\frac{2K^2(a,b) + G^2(a,b)}{G^4(a,b)} \end{aligned}$$

□

Proposition 6. *Let $0 < a < b < \infty$. Then one has the inequality*

$$A^{2p}(a.b) \leq 2\left[L_{2p}^p(a,b) + L_p^{2p}(a,b)\right] \tag{3.6}$$

Proof. If we choose in Theorem 11., $f:[a,b]\to[0,\infty)$, $f(x)=g(x)=x^p$ and $p\geq 2$ (which is convex) we get that:

$$\begin{aligned} \frac{1}{2}\left(\frac{a+b}{2}\right)^{2p}-\frac{1}{b-a}\int_a^b x^{2p}dx &\leq \frac{1}{(b-a)^2}\int_a^b x^p dx\int_a^b x^p dx \\ \frac{1}{2}A^{2p}(a.b)-L_{2p}^{p}(a,b) &\leq L_p^{2p}(a,b) \end{aligned}$$

□

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Department of Mathematics, Faculty of Science and Arts, 7 Aralik University, Kilis, 79000, Turkey.

E-mail address: mevluttunc@kilis.edu.tr