

Hermite-Hadamard's Type Inequalities for Convex Functions of Selfadjoint Operators in Hilbert Spaces

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ABSTRACT. Some Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications in relation with the celebrated Hölder-McCarthy's inequality for positive operators and Ky Fan's inequality for real numbers are given as well.

1. Introduction

If $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I , then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$(HH) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.$$

This remarkable result is well known in the literature as the *Hermite-Hadamard inequality* [17].

For various generalizations, extensions, reverses and related inequalities, see [1], [2], [9], [11], [13], [14], [15], [17] the monograph [8] and the references therein.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The *Gelfand map* establishes a $*$ -isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A , denoted $Sp(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [10, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- (i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- (ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- (iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
- (iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A .

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If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *1_H*.e. $f(A)$ is a positive operator on H . Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$(P) \quad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$.

Jensen's inequality for convex functions is one of the most important result in the Theory of Inequalities due to the fact that many other famous inequalities are particular cases of this for appropriate choices of the function involved, see for instance [20, p.].

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [21] (see also [10, p. 5]):

THEOREM 1 (Mond-Pečarić, 1993, [21]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$(MP) \quad f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$$

for each $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [10, p. 57]:

THEOREM 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$(1.1) \quad \langle f(A)x, x \rangle \leq \frac{M - \langle Ax, x \rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x \rangle - m}{M - m} \cdot f(M)$$

for each $x \in H$ with $\|x\| = 1$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [10] and the references therein. For other results, see [23], [16], [22] and [19]. For recent results, see [4], [5], [6] and [7].

The main aim of the present paper is to establish some Hermite-Hadamard's type inequalities for convex functions. Applications in relation with the celebrated Hölder-McCarthy's inequality for positive operators and Ky Fan's inequality for real numbers are given as well.

2. Some Inequalities for Convex Functions

The following inequality related to the Mond-Pečarić result also holds:

THEOREM 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$(2.1) \quad \frac{f(m) + f(M)}{2} \geq \left\langle \frac{f(A) + f((m+M)1_H - A)}{2}, x, x \right\rangle \\ \geq \frac{f(\langle Ax, x \rangle) + f(m+M - \langle Ax, x \rangle)}{2} \geq f\left(\frac{m+M}{2}\right)$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x \rangle \neq \frac{m+M}{2}$, then also

$$(2.2) \quad \frac{f(\langle Ax, x \rangle) + f(m + M - \langle Ax, x \rangle)}{2} \\ \geq \frac{2}{\frac{m+M}{2} - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m+M-\langle Ax, x \rangle} f(u) du \geq f\left(\frac{m+M}{2}\right).$$

PROOF. Since f is convex on $[m, M]$ then for each $u \in [m, M]$ we have the inequalities

$$(2.3) \quad \frac{M-u}{M-m} f(m) + \frac{u-m}{M-m} f(M) \geq f\left(\frac{M-u}{M-m} m + \frac{u-m}{M-m} M\right) = f(u)$$

and

$$(2.4) \quad \frac{M-u}{M-m} f(M) + \frac{u-m}{M-m} f(m) \geq f\left(\frac{M-u}{M-m} M + \frac{u-m}{M-m} m\right) \\ = f(M + m - u).$$

If we add these two inequalities we get

$$f(m) + f(M) \geq f(u) + f(M + m - u)$$

for any $u \in [m, M]$, which, by the property (P) applied for the operator A , produces the first inequality in (2.1).

By the Mond-Pečarić inequality (MP) we have

$$\langle f((m+M)1_H - A)x, x \rangle \geq f(m + M - \langle Ax, x \rangle),$$

which together with (MP) produces the second inequality in (2.1).

The third part follows by the convexity of f .

In order to prove (2.2), we use the Hermite-Hadamard inequality for the convex functions f and the choices $a = \langle Ax, x \rangle$ and $b = m + M - \langle Ax, x \rangle$.

The proof is complete. \square

REMARK 1. We observe that, from the inequality (2.1) we have the following inequality in the operator order of $B(H)$

$$(2.5) \quad \left[\frac{f(m) + f(M)}{2} \right] 1_H \geq \frac{f(A) + f((m+M)1_H - A)}{2} \geq f\left(\frac{m+M}{2}\right) 1_H,$$

where f is a convex function on $[m, M]$ and A a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.

The case of log-convex functions may be of interest for applications and therefore is stated in:

COROLLARY 1. If g is a log-convex function on $[m, M]$, then

$$(2.6) \quad \sqrt{g(m)g(M)} \geq \exp \left\langle \ln [g(A)g((m+M)1_H - A)]^{1/2} x, x \right\rangle \\ \geq \sqrt{g(\langle Ax, x \rangle)g(m + M - \langle Ax, x \rangle)} \geq g\left(\frac{m+M}{2}\right)$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if $x \in H$ with $\|x\| = 1$ and $\langle Ax, x \rangle \neq \frac{m+M}{2}$, then also

$$(2.7) \quad \sqrt{g(\langle Ax, x \rangle) g(m+M-\langle Ax, x \rangle)} \\ \geq \exp \left[\frac{2}{\frac{m+M}{2} - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{m+M-\langle Ax, x \rangle} \ln g(u) du \right] \geq g\left(\frac{m+M}{2}\right).$$

The following result also holds

THEOREM 4. *Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a convex function on $[m, M]$, then

$$(2.8) \quad f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \\ \leq \frac{1}{2} [f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) + f(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle)] \\ \leq \left\langle \frac{1}{2} [f((1-t)A + tB) + f(tA + (1-t)B)]x, x \right\rangle \\ \leq \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} f(m) + \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m} f(M)$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Moreover, we have the Hermite-Hadamard's type inequalities:

$$(2.9) \quad f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \leq \int_0^1 f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt \\ \leq \left\langle \left[\int_0^1 f((1-t)A + tB) dt \right] x, x \right\rangle \\ \leq \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} f(m) + \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m} f(M)$$

each $x \in H$ with $\|x\| = 1$.

In addition, if we assume that $B - A$ is a positive definite operator, then

$$(2.10) \quad f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \langle (B-A)x, x \rangle \\ \leq \int_{\langle Ax, x \rangle}^{\langle Bx, x \rangle} f(u) du \leq \langle (B-A)x, x \rangle \left\langle \left[\int_0^1 f((1-t)A + tB) dt \right] x, x \right\rangle \\ \leq \langle (B-A)x, x \rangle \left[\frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} f(m) + \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m} f(M) \right].$$

PROOF. It is obvious that for any $t \in [0, 1]$ we have $Sp((1-t)A + tB), Sp(tA + (1-t)B) \subseteq [m, M]$.

On making use of the Mond-Pečarić inequality (MP) we have

$$(2.11) \quad f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) \leq \langle f((1-t)A + tB)x, x \rangle$$

and

$$(2.12) \quad f(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle) \leq \langle f(tA + (1-t)B)x, x \rangle$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Adding (2.11) with (2.12) and utilising the convexity of f we deduce the first two inequalities in (2.8).

By the inequality (1.1) we also have

$$(2.13) \quad \langle f((1-t)A + tB)x, x \rangle \leq \frac{M - (1-t)\langle Ax, x \rangle - t\langle Bx, x \rangle}{M - m} \cdot f(m) \\ + \frac{(1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle - m}{M - m} \cdot f(M)$$

and

$$(2.14) \quad \langle f(tA + (1-t)B)x, x \rangle \leq \frac{M - t\langle Ax, x \rangle - (1-t)\langle Bx, x \rangle}{M - m} \cdot f(m) \\ + \frac{t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle - m}{M - m} \cdot f(M)$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Now, if we add the inequalities (2.13) with (2.14) and divide by two, we deduce the last part in (2.8).

Integrating the inequality over $t \in [0, 1]$, utilising the continuity property of the inner product and the properties of the integral of operator-valued functions we have

$$(2.15) \quad f\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \\ \leq \frac{1}{2} \left[\int_0^1 f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt + \int_0^1 f(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle) dt \right] \\ \leq \left\langle \frac{1}{2} \left[\int_0^1 f((1-t)A + tB) dt + \int_0^1 f(tA + (1-t)B) dt \right] x, x \right\rangle \\ \leq \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M - m} f(m) + \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M - m} f(M).$$

Since

$$\int_0^1 f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt = \int_0^1 f(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle) dt$$

and

$$\int_0^1 f((1-t)A + tB) dt = \int_0^1 f(tA + (1-t)B) dt$$

then, by (2.15), we deduce the inequality (2.9).

The inequality (2.10) follows from (2.9) by observing that for $\langle Bx, x \rangle > \langle Ax, x \rangle$ we have

$$\int_0^1 f((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt = \frac{1}{\langle Bx, x \rangle - \langle Ax, x \rangle} \int_{\langle Ax, x \rangle}^{\langle Bx, x \rangle} f(u) du$$

for each $x \in H$ with $\|x\| = 1$. \square

REMARK 2. We observe that, from the inequalities (2.8) and (2.9) we have the following inequalities in the operator order of $B(H)$

$$(2.16) \quad \frac{1}{2} [f((1-t)A + tB) + f(tA + (1-t)B)] \\ \leq f(m) \frac{M1_H - \frac{A+B}{2}}{M-m} + f(M) \frac{\frac{A+B}{2} - m1_H}{M-m},$$

where f is a convex function on $[m, M]$ and A, B are selfadjoint operator on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$.

The case of log-convex functions is as follows:

COROLLARY 2. If g is a log-convex function on $[m, M]$, then

$$(2.17) \quad g\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \\ \leq \sqrt{g((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle)g(t\langle Ax, x \rangle + (1-t)\langle Bx, x \rangle)} \\ \leq \exp\left\langle \frac{1}{2} [\ln g((1-t)A + tB) + \ln g(tA + (1-t)B)]x, x \right\rangle \\ \leq g(m) \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} g(M) \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m}$$

for any $t \in [0, 1]$ and each $x \in H$ with $\|x\| = 1$.

Moreover, we have the Hermite-Hadamard's type inequalities:

$$(2.18) \quad g\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right) \\ \leq \exp\left[\int_0^1 \ln g((1-t)\langle Ax, x \rangle + t\langle Bx, x \rangle) dt\right] \\ \leq \exp\left\langle \left[\int_0^1 \ln g((1-t)A + tB) dt\right]x, x \right\rangle \\ \leq g(m) \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} g(M) \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m}$$

for each $x \in H$ with $\|x\| = 1$.

In addition, if we assume that $B - A$ is a positive definite operator, then

$$(2.19) \quad g\left(\left\langle \frac{A+B}{2}x, x \right\rangle\right)^{\langle (B-A)x, x \rangle} \leq \exp\left[\int_{\langle Ax, x \rangle}^{\langle Bx, x \rangle} \ln g(u) du\right] \\ \leq \exp\left[\langle (B-A)x, x \rangle \left\langle \left[\int_0^1 \ln g((1-t)A + tB) dt\right]x, x \right\rangle\right] \\ \leq \left[g(m) \frac{M - \langle \frac{A+B}{2}x, x \rangle}{M-m} g(M) \frac{\langle \frac{A+B}{2}x, x \rangle - m}{M-m}\right]^{\langle (B-A)x, x \rangle}$$

for each $x \in H$ with $\|x\| = 1$.

From a different perspective we have the following result as well:

THEOREM 5. *Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\begin{aligned}
 (2.20) \quad f\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) &\leq \int_0^1 f((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) dt \\
 &\leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle \\
 &\leq \frac{1}{2} [\langle f(A)x, x \rangle + f(\langle By, y \rangle)] \\
 &\leq \frac{1}{2} [\langle f(A)x, x \rangle + \langle f(B)y, y \rangle]
 \end{aligned}$$

and

$$\begin{aligned}
 (2.21) \quad f\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) &\leq \left\langle f\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) x, x \right\rangle \\
 &\leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle
 \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

PROOF. For a convex function f and any $u, v \in [m, M]$ and $t \in [0, 1]$ we have the double inequality:

$$(2.22) \quad f\left(\frac{u+v}{2}\right) \leq \frac{1}{2} [f((1-t)u + tv) + f(tu + (1-t)v)] \leq \frac{1}{2} [f(u) + f(v)].$$

Utilising the second inequality in (2.22) we have

$$\begin{aligned}
 (2.23) \quad \frac{1}{2} [f((1-t)u + t\langle By, y \rangle) + f(tu + (1-t)\langle By, y \rangle)] \\
 \leq \frac{1}{2} [f(u) + f(\langle By, y \rangle)]
 \end{aligned}$$

for any $u \in [m, M]$, $t \in [0, 1]$ and $y \in H$ with $\|y\| = 1$.

Now, on applying the property (P) to the inequality (2.23) for the operator A we have

$$\begin{aligned}
 (2.24) \quad \frac{1}{2} [\langle f((1-t)A + t\langle By, y \rangle) x, x \rangle + \langle f(tA + (1-t)\langle By, y \rangle) x, x \rangle] \\
 \leq \frac{1}{2} [\langle f(A)x, x \rangle + f(\langle By, y \rangle)]
 \end{aligned}$$

for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

On applying the Mond-Pečarić inequality (MP) we also have

$$\begin{aligned}
 (2.25) \quad \frac{1}{2} [f((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) + f(t\langle Ax, x \rangle + (1-t)\langle By, y \rangle)] \\
 \leq \frac{1}{2} [\langle f((1-t)A + t\langle By, y \rangle 1_H) x, x \rangle + \langle f(tA + (1-t)\langle By, y \rangle 1_H) x, x \rangle]
 \end{aligned}$$

for any $t \in [0, 1]$ and $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, integrating over t on $[0, 1]$ the inequalities (2.24) and (2.25) and taking into account that

$$\begin{aligned} & \int_0^1 \langle f((1-t)A + t\langle By, y \rangle 1_H) x, x \rangle dt \\ &= \int_0^1 \langle f(tA + (1-t)\langle By, y \rangle 1_H) x, x \rangle dt \\ &= \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle \end{aligned}$$

and

$$\int_0^1 f((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) dt = \int_0^1 f(t\langle Ax, x \rangle + (1-t)\langle By, y \rangle) dt,$$

we obtain the second and the third inequality in (2.20).

Further, on applying the Jensen integral inequality for the convex function f we also have

$$\begin{aligned} & \int_0^1 f((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) dt \\ & \geq f\left(\int_0^1 [(1-t)\langle Ax, x \rangle + t\langle By, y \rangle] dt\right) \\ & = f\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, proving the first part of (2.20).

Now, on utilising the first part of (2.22) we can also state that

$$(2.26) \quad f\left(\frac{u + \langle By, y \rangle}{2}\right) \leq \frac{1}{2} [f((1-t)u + t\langle By, y \rangle) + f(tu + (1-t)\langle By, y \rangle)]$$

for any $u \in [m, M]$, $t \in [0, 1]$ and $y \in H$ with $\|y\| = 1$.

Further, on applying the property (P) to the inequality (2.26) and for the operator A we get

$$\begin{aligned} & \left\langle f\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) x, x \right\rangle \\ & \leq \frac{1}{2} [\langle f((1-t)A + t\langle By, y \rangle 1_H) x, x \rangle + \langle f(tA + (1-t)\langle By, y \rangle 1_H) x, x \rangle] \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which, by integration over t in $[0, 1]$ produces the second inequality in (2.21). The first inequality is obvious. \square

REMARK 3. *It is important to remark that, from the inequalities (2.20) and (2.21) we have the following Hermite-Hadamard's type results in the operator order of $B(H)$ and for the convex function $f : [m, M] \rightarrow \mathbb{R}$*

$$(2.27) \quad \begin{aligned} f\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) & \leq \int_0^1 f((1-t)A + t\langle By, y \rangle 1_H) dt \\ & \leq \frac{1}{2} [f(A) + f(\langle By, y \rangle 1_H)] \end{aligned}$$

for any $y \in H$ with $\|y\| = 1$ and any selfadjoint operators A, B with spectra in $[m, M]$.

In particular, we have from (2.27)

$$(2.28) \quad f\left(\frac{A + \langle Ay, y \rangle 1_H}{2}\right) \leq \int_0^1 f((1-t)A + t\langle Ay, y \rangle 1_H) dt \\ \leq \frac{1}{2} [f(A) + f(\langle Ay, y \rangle 1_H)]$$

for any $y \in H$ with $\|y\| = 1$ and

$$(2.29) \quad f\left(\frac{A + s1_H}{2}\right) \leq \int_0^1 f((1-t)A + ts1_H) dt \leq \frac{1}{2} [f(A) + f(s)1_H]$$

for any $s \in [m, M]$.

As a particular case of the above theorem we have the following refinement of the Mond-Pečarić inequality:

COROLLARY 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$(2.30) \quad f(\langle Ax, x \rangle) \leq \left\langle f\left(\frac{A + \langle Ax, x \rangle 1_H}{2}\right) x, x \right\rangle \\ \leq \left\langle \left[\int_0^1 f((1-t)A + t\langle Ax, x \rangle 1_H) dt \right] x, x \right\rangle \\ \leq \frac{1}{2} [\langle f(A)x, x \rangle + f(\langle Ax, x \rangle)] \leq \langle f(A)x, x \rangle.$$

Finally, the case of log-convex functions is as follows:

COROLLARY 4. *If g is a log-convex function on $[m, M]$, then*

$$(2.31) \quad g\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) \leq \exp\left[\int_0^1 \ln g((1-t)\langle Ax, x \rangle + t\langle By, y \rangle) dt\right] \\ \leq \exp\left\langle \left[\int_0^1 \ln g((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle \\ \leq \exp\left[\frac{1}{2} [\langle \ln g(A)x, x \rangle + \ln g(\langle By, y \rangle)]\right] \\ \leq \exp\left[\frac{1}{2} [\langle \ln g(A)x, x \rangle + \langle \ln g(B)y, y \rangle]\right]$$

and

$$(2.32) \quad g\left(\frac{\langle Ax, x \rangle + \langle By, y \rangle}{2}\right) \leq \exp\left\langle \ln g\left(\frac{A + \langle By, y \rangle 1_H}{2}\right) x, x \right\rangle \\ \leq \exp\left\langle \left[\int_0^1 \ln g((1-t)A + t\langle By, y \rangle 1_H) dt \right] x, x \right\rangle$$

and

$$\begin{aligned}
(2.33) \quad g(\langle Ax, x \rangle) &\leq \exp \left\langle \ln g \left(\frac{A + \langle Ax, x \rangle 1_H}{2} \right) x, x \right\rangle \\
&\leq \exp \left\langle \left[\int_0^1 \ln g((1-t)A + t\langle Ax, x \rangle 1_H) dt \right] x, x \right\rangle \\
&\leq \exp \left[\frac{1}{2} [\langle \ln g(A) x, x \rangle + \langle \ln g(\langle Ax, x \rangle) x, x \rangle] \right] \leq \exp \langle \ln g(A) x, x \rangle
\end{aligned}$$

respectively, for each $x \in H$ with $\|x\| = 1$ and A, B selfadjoint operators with spectra in $[m, M]$.

It is obvious that all the above inequalities can be applied for particular convex or log-convex functions of interest. However, we will restrict ourselves to only a few examples that are connected with famous results such as the Hölder-McCarthy inequality or the Ky Fan inequality.

3. Applications for Hölder-McCarthy's Inequality

We have the following important inequality in Operator Theory that is well known as the Hölder-McCarthy inequality:

THEOREM 6 (Hölder-McCarthy, 1967, [18]). *Let A be a selfadjoint positive operator on a Hilbert space H . Then*

- (i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
- (ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
- (iii) If A is invertible, then $\langle A^{-r} x, x \rangle \geq \langle Ax, x \rangle^{-r}$ for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

We can improve the above result as follows:

PROPOSITION 1. *Let A be a selfadjoint positive operator on a Hilbert space H . If $r > 1$, then*

$$\begin{aligned}
(3.1) \quad \langle Ax, x \rangle^r &\leq \left\langle \left(\frac{A + \langle Ax, x \rangle 1_H}{2} \right)^r x, x \right\rangle \\
&\leq \left\langle \left[\int_0^1 ((1-t)A + t\langle Ax, x \rangle 1_H)^r dt \right] x, x \right\rangle \\
&\leq \frac{1}{2} [\langle A^r x, x \rangle + \langle Ax, x \rangle^r] \leq \langle A^r x, x \rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

If $0 < r < 1$, then the inequalities reverse in (3.1).

If A is invertible and $r > 0$, then

$$\begin{aligned}
(3.2) \quad \langle Ax, x \rangle^{-r} &\leq \left\langle \left(\frac{A + \langle Ax, x \rangle 1_H}{2} \right)^{-r} x, x \right\rangle \\
&\leq \left\langle \left[\int_0^1 ((1-t)A + t\langle Ax, x \rangle 1_H)^{-r} dt \right] x, x \right\rangle \\
&\leq \frac{1}{2} [\langle A^{-r} x, x \rangle + \langle Ax, x \rangle^{-r}] \leq \langle A^{-r} x, x \rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Follows from the inequality (2.31) applied for the power function.

Since the function $g(t) = t^{-r}$ for $r > 0$ is log-convex, then by utilising the inequality (2.33) we can improve the Hölder-McCarthy inequality as follows:

PROPOSITION 2. *Let A be a selfadjoint positive operator on a Hilbert space H . If A is invertible, then*

$$(3.3) \quad \begin{aligned} \langle Ax, x \rangle^{-r} &\leq \exp \left\langle \ln \left(\frac{A + \langle Ax, x \rangle 1_H}{2} \right)^{-r} x, x \right\rangle \\ &\leq \exp \left\langle \left[\int_0^1 \ln ((1-t)A + t\langle Ax, x \rangle 1_H)^{-r} dt \right] x, x \right\rangle \\ &\leq \exp \left[\frac{1}{2} \left[\langle \ln A^{-r} x, x \rangle + \ln \langle Ax, x \rangle^{-r} \right] \right] \leq \exp \langle \ln A^{-r} x, x \rangle \end{aligned}$$

for all $r > 0$ and $x \in H$ with $\|x\| = 1$.

Now, from a different perspective, we can state the following operator power inequalities:

PROPOSITION 3. *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M] \subset [0, \infty)$, then*

$$(3.4) \quad \begin{aligned} \frac{m^r + M^r}{2} &\geq \left\langle \frac{A^r + ((m+M)1_H - A)^r}{2} x, x \right\rangle \\ &\geq \frac{\langle Ax, x \rangle^r + (m+M - \langle Ax, x \rangle)^r}{2} \geq \left(\frac{m+M}{2} \right)^r \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$ and $r > 1$.

If $0 < r < 1$ then the inequalities reverse in (3.4).

If A is positive definite and $r > 0$, then

$$(3.5) \quad \begin{aligned} \frac{m^{-r} + M^{-r}}{2} &\geq \left\langle \frac{A^{-r} + ((m+M)1_H - A)^{-r}}{2} x, x \right\rangle \\ &\geq \frac{\langle Ax, x \rangle^{-r} + (m+M - \langle Ax, x \rangle)^{-r}}{2} \geq \left(\frac{m+M}{2} \right)^{-r} \end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

The proof follows by the inequality (2.1).

Finally we have:

PROPOSITION 4. *Assume that A and B are selfadjoint operators with spectra in $[m, M] \subset [0, \infty)$ and $x \in H$ with $\|x\| = 1$ and such that $\langle Ax, x \rangle \neq \langle Bx, x \rangle$.*

If $r > 1$ or $r \in (\infty, -1) \cup (-1, 0)$ then we have

$$(3.6) \quad \begin{aligned} \left\langle \left(\frac{A+B}{2} \right) x, x \right\rangle^r &\leq \frac{1}{r+1} \cdot \frac{\langle Ax, x \rangle^{r+1} - \langle Bx, x \rangle^{r+1}}{\langle Ax, x \rangle - \langle Bx, x \rangle} \\ &\leq \left\langle \left[\int_0^1 ((1-t)A + tB)^r dt \right] x, x \right\rangle \\ &\leq \frac{M - \langle \frac{A+B}{2} x, x \rangle}{M-m} m^r + \frac{\langle \frac{A+B}{2} x, x \rangle - m}{M-m} M^r. \end{aligned}$$

If $0 < r < 1$, then the inequalities reverse in (3.6).
If A and B are positive definite, then

$$(3.7) \quad \left\langle \left(\frac{A+B}{2} \right) x, x \right\rangle^{-1} \leq \frac{\ln \langle Bx, x \rangle - \ln \langle Ax, x \rangle}{\langle Bx, x \rangle - \langle Ax, x \rangle} \\ \leq \left\langle \left[\int_0^1 ((1-t)A + tB)^{-1} dt \right] x, x \right\rangle \\ \leq \frac{M - \langle \frac{A+B}{2} x, x \rangle}{(M-m)m} + \frac{\langle \frac{A+B}{2} x, x \rangle - m}{(M-m)M}.$$

4. Applications for Ky Fan's Inequality

Consider the function $g : (0, 1) \rightarrow \mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r$, $r > 0$. Observe that for the new function $f : (0, 1) \rightarrow \mathbb{R}$, $f(t) = \ln g(t)$ we have

$$f'(t) = \frac{-r}{t(1-t)} \text{ and } f''(t) = \frac{2r\left(\frac{1}{2}-t\right)}{t^2(1-t)^2} \text{ for } t \in (0, 1)$$

showing that the function g is log-convex on the interval $(0, \frac{1}{2})$.

If $p_i > 0$ for $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $t_i \in (0, \frac{1}{2})$ for $i \in \{1, \dots, n\}$, then by applying the Jensen inequality for the convex function f (with $r = 1$) on the interval $(0, \frac{1}{2})$ we get

$$(4.1) \quad \frac{\sum_{i=1}^n p_i t_i}{1 - \sum_{i=1}^n p_i t_i} \geq \prod_{i=1}^n \left(\frac{t_i}{1-t_i} \right)^{p_i},$$

which is the weighted version of the celebrated *Ky Fan's inequality*, see [3, p. 3].

This inequality is equivalent with

$$\prod_{i=1}^n \left(\frac{1-t_i}{t_i} \right)^{p_i} \geq \frac{1 - \sum_{i=1}^n p_i t_i}{\sum_{i=1}^n p_i t_i},$$

where $p_i > 0$ for $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n p_i = 1$ and $t_i \in (0, \frac{1}{2})$ for $i \in \{1, \dots, n\}$.

By the weighted arithmetic mean - geometric mean inequality we also have that

$$\sum_{i=1}^n p_i (1-t_i) t_i^{-1} \geq \prod_{i=1}^n \left(\frac{1-t_i}{t_i} \right)^{p_i}$$

giving the double inequality

$$(4.2) \quad \sum_{i=1}^n p_i (1-t_i) t_i^{-1} \geq \prod_{i=1}^n ((1-t_i) t_i^{-1})^{p_i} \geq \sum_{i=1}^n p_i (1-t_i) \left(\sum_{i=1}^n p_i t_i \right)^{-1}.$$

PROPOSITION 5. Let A be a selfadjoint positive operator on a Hilbert space H . If A is invertible and $Sp(A) \subset (0, \frac{1}{2})$, then

$$\begin{aligned}
(4.3) \quad & \left(\langle (1_H - A)x, x \rangle \langle Ax, x \rangle^{-1} \right)^r \\
& \leq \exp \left\langle \ln \left([1_H - A + \langle (1_H - A)x, x \rangle 1_H] (A + \langle Ax, x \rangle 1_H)^{-1} \right)^r x, x \right\rangle \\
& \leq \left\langle \exp \left[\int_0^1 \ln \left((1-t)(1_H - A) + t \langle (1_H - A)x, x \rangle 1_H \right) \right. \right. \\
& \quad \left. \left. \times ((1-t)A + t \langle Ax, x \rangle 1_H)^{-1} \right]^r dt \right\rangle x, x \right\rangle \\
& \leq \exp \left[\frac{1}{2} \left[\left\langle \ln [(1_H - A)A^{-1}]^r x, x \right\rangle + \ln \left(\langle (1_H - A)x, x \rangle \langle Ax, x \rangle^{-1} \right)^r \right] \right] \\
& \leq \exp \left\langle \ln [(1_H - A)A^{-1}]^r x, x \right\rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

It follows from the inequality (2.33) applied for the log-convex function $g : (0, 1) \rightarrow \mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r$, $r > 0$.

PROPOSITION 6. Assume that A is a selfadjoint operator with $Sp(A) \subset (0, \frac{1}{2})$ and $s \in (0, \frac{1}{2})$. Then we have the following inequality in the operator order of $B(H)$:

$$\begin{aligned}
(4.4) \quad & \ln \left[[(2-s)1_H - A] (A + s1_H)^{-1} \right] \\
& \leq \int_0^1 \ln \left([(1-ts)1_H - (1-t)A] ((1-t)A + ts1_H)^{-1} \right) dt \\
& \leq \frac{1}{2} \left(\ln [(1_H - A)A^{-1}]^r + \ln \left(\frac{1-s}{s} \right)^r 1_H \right).
\end{aligned}$$

It follows from the inequality (2.29) applied for the log-convex function $g : (0, 1) \rightarrow \mathbb{R}$, $g(t) = \left(\frac{1-t}{t}\right)^r$, $r > 0$.

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