.

IOACHIMESCU'S CONSTANT

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ABSTRACT. The Ioachimescu's constant $\mathcal{I} = 0.539645\cdots$ is defined as the limit of the sequence $I_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n} - 1) \quad (n \in \mathbb{N})$. In this paper, we establish the asymptotic representation for the sequence $(I_n)_{n\in\mathbb{N}}$, and obtain the upper and lower bounds of the Ioachimescu's constant.

1. INTRODUCTION

A. G. Ioachimescu [2] proposed in 1895 a problem in which he asked to be shown that the sequence $(S_n)_{n\in\mathbb{N}}$, defined by

$$S_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n} \quad \text{for} \quad n \in \mathbb{N},$$

is convergent and its limit lies between -2 and -1.

Many generalizations and other results regarding the above-mentioned problem have been obtained by Alina Sîntămărian (see, for example, [3, pp. 27-33], [4, 5, 6, 7]; see also various closely-related references cited therein). Sîntămărian [7] considered the sequence $(I_n)_{n\in\mathbb{N}}$ defined by

$$I_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2(\sqrt{n} - 1),$$

and proved that

$$\frac{1}{2\sqrt{n + \frac{1}{5}}} < I_n - \mathcal{I} < \frac{1}{2\sqrt{n + \frac{1}{6}}} \quad \text{for} \quad n \in \mathbb{N}, \tag{1}$$

where $\mathcal{I}$ denotes the limit of $(I_n)_{n\in\mathbb{N}}$, calling it Ioachimescu's constant. The inequalities (1) implies $\mathcal{I} = 0.539645\cdots$.

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In this paper, we establish the asymptotic representation for the sequence $(I_n)_{n\in\mathbb{N}}$, and obtain the upper and lower bounds of the Ioachimescu's constant.

Theorem 1. *For all integers $n \geq 1$ and $p \geq 0$, then the following asymptotic representation holds:*

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) = \mathcal{I} + \frac{1}{2\sqrt{n}} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \frac{1}{n^{2k-\frac{1}{2}}} + O(1/n^{2p+\frac{3}{2}}), \quad (2)$$

where the B_{2k} are the Bernoulli numbers defined by

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Remark 1. (2) *can be written as*

$$\mathcal{I} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) - \frac{1}{2\sqrt{n}} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \frac{1}{n^{2k-\frac{1}{2}}} + O(1/n^{2p+\frac{3}{2}}). \quad (3)$$

First three Bernoulli numbers with even indices are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$

and then, we have

$$\mathcal{I} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) - \frac{1}{2\sqrt{n}} + \frac{1}{24n^{3/2}} - \frac{1}{384n^{7/2}} + \frac{1}{1024n^{11/2}} + O\left(\frac{1}{n^{15/2}}\right).$$

Theorem 2. *For all integers $p \geq 0$, then*

$$\frac{1}{2} + \sum_{k=1}^{2p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} < \mathcal{I} < \frac{1}{2} + \sum_{k=1}^{2p+1} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}}. \quad (4)$$

It is well-known that $(-1)^{k-1}B_{2k} > 0$. In view of (2), we pose the following conjecture 1.

Conjecture 1. *For all integers $n \geq 1$ and $p \geq 0$, then*

$$\frac{1}{2\sqrt{n}} - \sum_{k=1}^{2p+1} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \frac{1}{n^{2k-\frac{1}{2}}} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) - \mathcal{I}$$
$$< \frac{1}{2\sqrt{n}} - \sum_{k=1}^{2p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \frac{1}{n^{2k-\frac{1}{2}}}.$$

Define the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_n = \frac{1}{4\left[\sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) - \mathcal{I}\right]^2} - n.$$

Straightforward calculation produces

$$x_1 = 0.17965\cdots, \quad x_2 = 0.17496\cdots, \quad x_3 = 0.17265\cdots.$$

From the asymptotic formula

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n}-1) - \mathcal{I} = \frac{1}{2\sqrt{n}} - \frac{1}{24n^{3/2}} + O\left(\frac{1}{n^{7/2}}\right),$$

we conclude that

$$x_n = \frac{\frac{1}{6} + O(n^{-1})}{1 + O(n^{-1})} \to \frac{1}{6} \quad (n \to \infty).$$

Hence, it is natural to propose the following conjecture 2.

Conjecture 2. *For all integers $n \geq 1$, then*

$$\frac{1}{2\sqrt{n+a}} \leq I_n - \mathcal{I} < \frac{1}{2\sqrt{n+b}}$$

with the best possible constants

$$a = \frac{1}{4(1-\mathcal{I})^2} - 1 = 0.17965\cdots \quad \text{and} \quad b = \frac{1}{6} = 0.16666\cdots.$$

2. Proofs of Theorems 1 and 2

We prove Theorem 1 by using the Euler-Maclaurin formula. Recall that let f a function of class C^{2p+2} on an interval $[a, b]$ and let $h = (b-a)/m$ a subdivision of this interval into m equal parts then we have the important Euler-Maclaurin formula: There exist $0 < \theta < 1$ and

$$\begin{aligned}\sum_{k=0}^{m} f(a+kh) = {} & \frac{1}{h}\int_a^b f(x)\,\mathrm{d}x + \frac{f(a)+f(b)}{2} \\ & + \sum_{k=1}^{p} \frac{h^{2k-1}}{(2k)!} B_{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) \\ & + \frac{h^{2p+2}}{(2p+2)!} B_{2p+2} \sum_{k=0}^{m-1} f^{(2p+2)}(a+kh+\theta h).\end{aligned} \tag{5}$$

This formula was first studied by Euler in 1732 and independently by Maclaurin in 1742. Euler used it to compute sums of slow converging series and Maclaurin used it as a numerical quadrature formula.

With the same conditions, setting $m = n - 1, a = 1, b = n, h = 1$ (5) becomes:

$$\begin{aligned}\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}x + \frac{f(1) + f(n)}{2} \\ + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(1)\big) + R_n(f,p),\end{aligned} \tag{6}$$

where

$$R_n(f,p) = \frac{B_{2p+2}}{(2p+2)!} \sum_{k=1}^{n-1} f^{(2p+2)}(k+\theta) \tag{7}$$

is the remainder bounded by

$$R_n(f,p) \le \frac{2}{(2\pi)^{2p}} \int_1^n \left|f^{(2p+1)}(x)\right| \mathrm{d}x.$$

Proof of Theorem 1. Setting $f(x) = \frac{1}{\sqrt{x}}$, we have

$$f^{(n)}(x) = (-1)^n \frac{(2n-1)!!}{2^n x^{n+\frac{1}{2}}} \quad \text{and} \quad f^{(2k-1)}(x) = -\frac{(4k-3)!!}{2^{2k-1} x^{2k-\frac{1}{2}}}.$$

Euler-Maclaurin formula yields for a given p:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n} - 1) = \frac{1}{2} + \frac{1}{2\sqrt{n}} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \left(1 - \frac{1}{n^{2k-\frac{1}{2}}}\right) + R_n(f,p). \tag{8}$$

When $n \to \infty$, the left-hand side of the equality (8) tends to $\mathcal{I}$ and the equality gives

$$\mathcal{I} = \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} + R_\infty(f,p), \tag{9}$$

finally

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2(\sqrt{n} - 1) = \mathcal{I} + \frac{1}{2\sqrt{n}} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \frac{(4k-3)!!}{2^{2k-1}} \frac{1}{n^{2k-\frac{1}{2}}} \\ + \big(R_n(f,p) - R_\infty(f,p)\big).\end{aligned}$$

Check that

$$
\begin{aligned}
& n^{2p-\frac{1}{2}}\left|R_n(f,p)-R_\infty(f,p)\right| \\
&= \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}n^{2p-\frac{1}{2}}\sum_{k=n}^{\infty}\frac{1}{(k+\theta)^{2p+\frac{5}{2}}} \\
&< \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}n^{2p-\frac{1}{2}}\sum_{k=n}^{\infty}\frac{1}{k^{2p+\frac{5}{2}}} \\
&= \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}\frac{1}{\sqrt{n}}\left[\frac{1}{n^{5/2}}+\left(\frac{n}{n+1}\right)^{2p}\frac{1}{(n+1)^{5/2}}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.+\left(\frac{n}{n+2}\right)^{2p}\frac{1}{(n+2)^{5/2}}+\cdots\right] \\
&< \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}\frac{1}{\sqrt{n}}\left[\frac{1}{n^{5/2}}+\frac{1}{(n+1)^{5/2}}+\frac{1}{(n+2)^{5/2}}+\cdots\right] \\
&= \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}\frac{1}{\sqrt{n}}\sum_{k=n}^{\infty}\frac{1}{k^{5/2}} \\
&< \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}\frac{1}{\sqrt{n}}\sum_{k=1}^{\infty}\frac{1}{k^{5/2}} \\
&= \frac{|B_{2p+2}|}{(2p+2)!}\frac{(4p+3)!!}{2^{2p+2}}\frac{\zeta(\frac{5}{2})}{\sqrt{n}}\to 0 \quad (n\to\infty).
\end{aligned}
$$

Hence, (2) holds. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $f(x)=\frac{1}{\sqrt{x}}$. From (8), we get

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2(\sqrt{n}-1) &= \frac{1}{2}+\frac{1}{2\sqrt{n}}+\sum_{k=1}^{2p}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\left(1-\frac{1}{n^{2k-\frac{1}{2}}}\right) \\
&\quad +\frac{B_{4p+2}}{(4p+2)!}\sum_{k=1}^{n-1}\frac{(8p+3)!!}{2^{4p+2}(k+\theta)^{4p+\frac{5}{2}}}.
\end{aligned}
\tag{10}
$$

Since $B_{4p+2}>0$, we get from (10):

$$
\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2(\sqrt{n}-1) > \frac{1}{2}+\frac{1}{2\sqrt{n}}+\sum_{k=1}^{2p}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\left(1-\frac{1}{n^{2k-\frac{1}{2}}}\right). \tag{11}
$$

Let $n\to\infty$, from (11) we get the left-hand side of inequality (4).

From (8), we get

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2(\sqrt{n}-1)=\frac{1}{2}+\frac{1}{2\sqrt{n}}+\sum_{k=1}^{2p+1}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\left(1-\frac{1}{n^{2k-\frac{1}{2}}}\right)$$
$$+\frac{B_{4p+4}}{(4p+4)!}\sum_{k=1}^{n-1}\frac{(8p+7)!!}{2^{4p+4}(k+\theta)^{4p+\frac{9}{2}}}. \tag{12}$$

Since $B_{4p+4}<0$, we get from (12):

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k}}-2(\sqrt{n}-1)<\frac{1}{2}+\frac{1}{2\sqrt{n}}+\sum_{k=1}^{2p+1}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\left(1-\frac{1}{n^{2k-\frac{1}{2}}}\right). \tag{13}$$

Let $n\to\infty$, from (13) we get the right-hand side of inequality (4). The proof of Theorem 2 is complete. □

3. A similar sequence

Now define the sequence $(J_n)_{n\in\mathbb{N}}$ by

$$J_n=\sum_{k=1}^{n}\frac{1}{\sqrt{k+\frac{1}{2}}}-2\left(\sqrt{n+\frac{1}{2}}-\sqrt{\frac{3}{2}}\right). \tag{14}$$

The same arguments which were used on $(I_n)_{n\in\mathbb{N}}$ can be used on $(J_n)_{n\in\mathbb{N}}$ to give the following results.

Theorem 3. *Let the sequence $(J_n)_{n\in\mathbb{N}}$ be defined by (14). Then, the following results hold:*

(i) *For all integers $n\geq 1$ and $p\geq 0$, then*

$$J_n=\mathcal{J}+\frac{1}{2\sqrt{n+\frac{1}{2}}}-\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}\frac{1}{(n+\frac{1}{2})^{2k-\frac{1}{2}}}+O\left(\frac{1}{(n+\frac{1}{2})^{2p+\frac{3}{2}}}\right), \tag{15}$$

where $\mathcal{J}=0.4304\cdots$ is the limit of the sequence $(J_n)_{n\in\mathbb{N}}$.

(ii) *For all integers $p\geq 0$, then*

$$\frac{1}{\sqrt{6}}+\sum_{k=1}^{2p}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}<\mathcal{J}<\frac{1}{\sqrt{6}}+\sum_{k=1}^{2p+1}\frac{B_{2k}}{(2k)!}\frac{(4k-3)!!}{2^{2k-1}}. \tag{16}$$

The table below shows that the sequences I_n and J_n are both strictly decreasing.

n	I_n	J_n
1	1	0.8164965809277
2	0.8786796564403	0.7361641955762
3	0.8203554352384	0.6913069527954
4	0.7844570503761	0.6617281732411
5	0.7595346908765	0.6403545332482
6	0.7409394507736	0.6239830497552
7	0.7263807872200	0.6109253599664
8	0.7145825504501	0.6001962104578
9	0.7047701332758	0.5911769445957
10	0.6964425789559	0.5834569490810
100	0.5896038247841	0.4802116479412
1000	0.5554555618755	0.4461836572959
10000	0.5446454495240	0.4353773703310
100000	0.5412266286996	0.4319586705380
1000000	0.5401454911987	0.4308775368413
10000000	0.5398036034421	0.4305356512959
100000000	0.5396954930401	0.4304275334652

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